FINITE DEFORMATION: GENERAL THEORY

The notes on finite deformation have been divided into three parts:

- General theory (http://imechanica.org/node/538)
- Elasticity of rubber-like materials (http://imechanica.org/node/14146)
- Special cases (http://imechanica.org/node/5065)

The three parts can be read in any order.

DIVISION OF LABOR

A body is made of atoms. Each atom is made of electrons, protons and neutrons. Each proton is made of... This kind of description is good for studying the fundamental nature of matter, but not for many other purposes. We will not go very far in life if we keep picturing a bridge as a pile of atoms. Instead, we will develop a different description, called continuum mechanics. Continuum mechanics studies how a force causes a body to deform. Continuum mechanics is effective whenever we can identify two widely separated length scales.

Two length scales. The deformation of the body is in general inhomogeneous—that is, the amount of deformation varies from one part of the body to another part. When we examine the deformation of a body, we can identify two length scales:

- length scale over which the macroscopic variation of deformation occurs
- length scale over which the microscopic process of deformation occurs.

For example, when a rubber eraser is bent, the macroscopic deformation varies over a length scaled with the thickness of the eraser (several millimeters). The rubber is a network of molecular chains. The microscopic process of deformation occurs over the length scaled with the length of an individual molecular chain (several nanometers).

Representative elementary volume. In many applications, the two length scales are widely separated. If they are, we can describe the behavior of the material by using a volume much larger than the size characteristic of the microscopic process of deformation, but much smaller than the size characteristic of the macroscopic deformation. Such a volume is known as a representative elementary volume (REV).

In the rubber eraser, for example, the microscopic process is the thermal motion of individual polymer chains, the body is the whole eraser, and the REV can be a small piece of the eraser. This piece is still large compared to individual polymer chains.

As another example, consider an airplane wing made of aluminum. The microscopic process can be activities of dislocations in aluminum, the body can be the entire wing, and the REV can be a tensile specimen of the aluminum. The tensile specimen is much smaller than the wing, but much larger than the individual grains of the aluminum.

The size of REV should be selected well between the two lengths scales. If a volume is too small, the volume cannot be treated as a continuum. If a volume is too large, the shape of the body affects the behavior of the volume.

Exercise. Describe the separation of length scales in each of the three types of materials: glasses, metals, and fiber-reinforced composites. In each type, how large does the representative elementary volume need to be?

Exercise. Continuum mechanics is sometimes applied to analyze living cells. Can we really separate the two length scales? What do we mean by representative elementary volume?

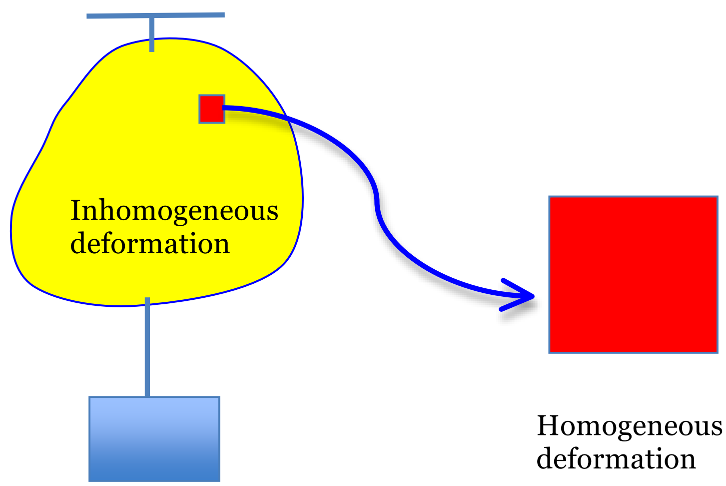
When a body deforms, does each small piece preserve its identity? We have tacitly assumed that, when a body deforms, each small piece in the body preserves its identity. Whether this assumption is valid can be determined by experiments. For example, we can paint a grid on the body. After deformation, if the grid is distorted but remains intact, then we say that the deformation preserves the identity of each small piece. If, however, after the deformation the grid disintegrates, we should not assume that the deformation preserves the identity of each small piece.

Whether a deformation of the body preserves the identity of a small piece in the body depends on the size of the piece and the time scale over which we observe it. A rubber, for example, consists of crosslinked long-chain molecules. If our grid is over a size much larger than the individual molecular chains, then deformation will not cause the grid to disintegrate. By contrast, a liquid consists of molecules that can change neighbors. A grid painted on a body of liquid, no matter how coarse the grid is, will disintegrate over a long enough time. Similar remarks may be made for metals undergoing plastic deformation. Also, in many situations, the body will grow over time. Examples include growth of cells in a tissue, and growth of thin films when atoms diffuse into the films. The combined growth and deformation clearly does not preserve the identity of each small piece of the body.

In these notes, we will assume that the identity of each small piece is preserved as the body deforms.

Division of labor. To analyze the inhomogeneous deformation in the body, the continuum theory regards the body as a sum of many pieces. Each piece evolves in time through a sequence of homogeneous deformations. All the pieces are then put together to represent the inhomogeneous deformation of the entire body. The division of labor results in two levels of analysis:

- Homogeneous deformation of a piece.
- Inhomogeneous deformation of a body.



Each level of analysis requires three ingredients:

- Geometry of deformation
- Balance of forces
- Model of material

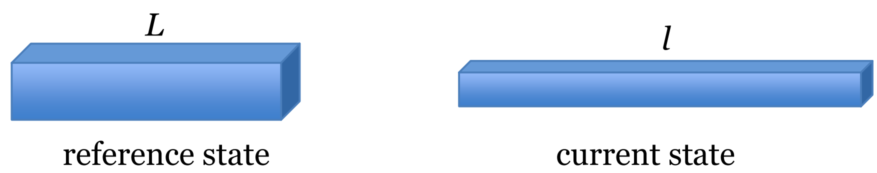
Continuum mechanics expresses the three ingredients into mathematical forms, which you will learn in this course.

HOMOGENEOUS DEFORMATION

Stretch of a rod. A rod deforms from one state to another state. The two states are called, respectively, the reference state and the current state. The length of the rod is L in the reference state, and l in the current state. The ratio of the two lengths defines the stretch of the rod:

$$\lambda = \frac{l}{L}.$$

Whenever convenient, we follow the convention of using the uppercase of a letter to label a quantity in the reference state, and using the lowercase of the same letter to label the quantity in the current state.

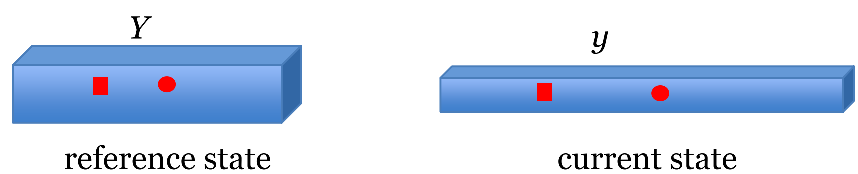


The above definition uses the lengths of the entire rod in the two states, and is valid even when the deformation of the rod is inhomogeneous. How do we find out if the deformation of the rod is homogeneous or inhomogeneous? We mark two material particles along the axis of the rod. We measure the distance Y between the two particles when the rod is in the reference state, and then measure the distance y between the two particles when the rod is in the current state. If the ratio y/Y is the same for any choices of the two material particles, the deformation of the rod is homogeneous.

We write

$$y = \lambda Y.$$

The deformation of the rod is said to be homogeneous if λ is the same for any choices of the two material particles along the axis of the rod.



We can also mark equally spaced and parallel lines on the surface of the undeformed rod. When we stretch the rod, the spacing between any two lines increases, but all lines are still equally spaced and parallel. Stretching of the rod causes homogenous deformation.

If we bend the rod, the lines will no longer be parallel. Bending of the rod causes inhomogeneous deformation.

Deformation of a body. Now consider a body undergoing a general state of homogenous deformation in the three-dimensional space. When the body deforms from a reference state to a current state, how do we find out if the deformation is homogeneous?

A homogeneous deformation of the body is described as follows. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms three families of parallel, uniformly-spaced lines. When the body is in the current state, the set of material particles still forms three families of parallel, uniformly-spaced lines. The homogeneous deformation may change the spacing between, and the orientation of, each family of the straight lines.

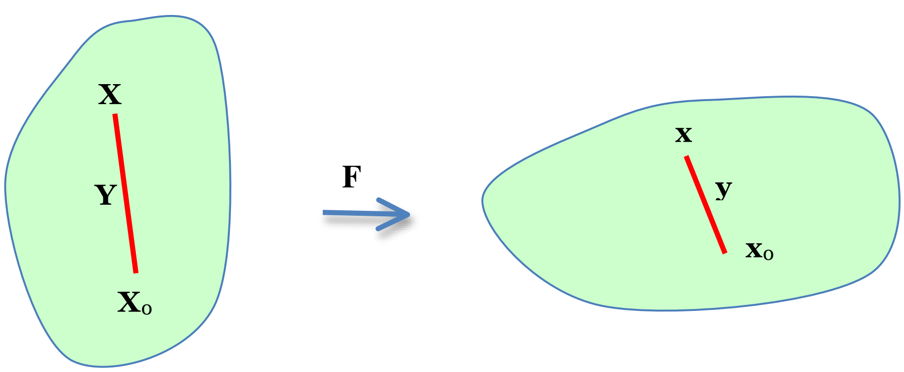
Deformation gradient. We call each small part of the body a material particle, and each small part of the space a place. As the body deforms, each material particle moves from one place to another place in the space, forming a trajectory.

Each place in the space has three coordinates. When the body is in the reference state, a material particle occupies a place in the space, and the coordinates of the place are written $\mathbf{X}$. When the body is in the current state, the same material particle occupies a different place in space, and the coordinates of the space are written $\mathbf{x}$.

We mark two material particles in the body. When the body is in the reference state, the two material particles are at places $\mathbf{X}_o$ and $\mathbf{X}$, and they are the two ends of a vector, $\mathbf{X}-\mathbf{X}_o$. When the body is in the current state, the same two material particles are at places $\mathbf{x}_o$ and $\mathbf{x}$, and they are the two ends of another vector, $\mathbf{x}-\mathbf{x}_o$. The deformation of the body maps the material vector in the reference state to the material vector in the current state:

$$\mathbf{x}-\mathbf{x}_o = \mathbf{F}\left(\mathbf{X}-\mathbf{X}_o\right).$$

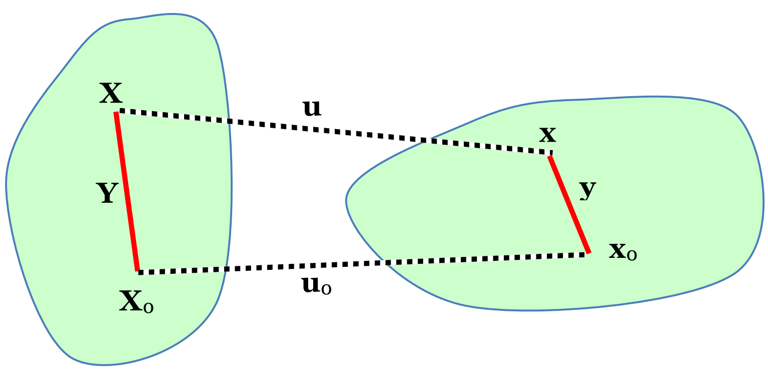
The operator $\mathbf{F}$ is known as the *deformation gradient*. The deformation of the body is homogeneous if $\mathbf{F}$ is a linear operator, and is the same for any choices of the two material particles. These words are too abstract, and we will describe more in the following pages.



reference state current state

The above expression calculates the place $\mathbf{x}$ of any material particle when the body is in the current state, provided we know (i) the place $\mathbf{X}$ of the same material particle when the body is in the reference state, (ii) the coordinate $\mathbf{X}_o$ of one particular material particle when the body is in the reference state, (iii) the coordinate $\mathbf{x}_o$ of this material particle when the body is in the current state, and (iv) the deformation gradient $\mathbf{F}$.

Translation, rotation and stretch. When a material particle moves from a place $\mathbf{x}_o$ in the reference state to a place $\mathbf{X}_o$ in the current state, the displacement of the material particle is $\mathbf{u}_o = \mathbf{x}_o - \mathbf{X}_o$. When another material particle moves from a place $\mathbf{x}$ in the reference state to a place $\mathbf{X}$ in the current state, the displacement of the material particle is $\mathbf{u} = \mathbf{x} - \mathbf{X}$.



reference state current state

Rewrite $\mathbf{x} - \mathbf{x}_o = \mathbf{F}(\mathbf{X} - \mathbf{X}_o)$ as

$$\mathbf{u} - \mathbf{u}_o = (\mathbf{F} - \mathbf{I})(\mathbf{X} - \mathbf{X}_o),$$

where $\mathbf{I}$ is the identity tensor. In the special case when the homogeneous deformation of the body is a rigid-body translation, the deformation gradient is the identity operator, $\mathbf{F} = \mathbf{I}$, and the two material particles have an identical displacement, $\mathbf{u} = \mathbf{u}_o$.

In the general case, however, the homogeneous deformation also rotates and stretches the body, $\mathbf{F} \neq \mathbf{I}$, and the two material particles have different displacements, $\mathbf{u} \neq \mathbf{u}_o$. The homogeneous deformation consists of three types: translation, rotation and stretch. The deformation translates a particular material particle from the place $\mathbf{X}_o$ in the reference state to the place $\mathbf{x}_o$ in the current state. After this translation, with the position of the particular material particle fixed at the place $\mathbf{x}_o$ in space, we stretch and rotate the body using $\mathbf{F}$. We will describe stretch and rotation later.

Material segment. When the body is in the reference state, we mark a set of material particles that forms a segment of a straight line. As the body undergoes a homogeneous deformation, the set of material particles behaves like a rod: it translates, rotates and stretches. The homogeneous deformation of the body, however, does not bend the segment. When the body is in the current state, the same set of material particles still forms a segment of a straight line.

We represent the set of material particles by a vector $\mathbf{Y}$ in the reference state, and by a vector $\mathbf{y}$ in the current state. For example, consider the two material particles. When the body is in the reference state, the two material particles are at places $\mathbf{X}_o$ and $\mathbf{X}$, and they are the two ends of the vector, $\mathbf{Y} = \mathbf{X} - \mathbf{X}_o$. When the body is in the current state, the same two material particles are at the places $\mathbf{x}_o$ and $\mathbf{x}$, and they are the two ends of another vector, $\mathbf{y} = \mathbf{x} - \mathbf{x}_o$.

The above geometric picture corresponds the algebraic formula:

$$\mathbf{y} = \mathbf{F}(\mathbf{Y}).$$

Thus, the deformation gradient maps a material segment in the reference state to the same material segment in the current state.

Compare the two definitions, $l = \lambda L$ and $\mathbf{y} = \mathbf{F}(\mathbf{Y})$. We replace the length L of the rod in the reference state with the vector $\mathbf{Y}$, replace the length l of the rod in the current state with the vector $\mathbf{y}$, and replace the stretch λ of the rod is with the operator $\mathbf{F}$. Just as the stretch is a measure of a homogeneous deformation of a rod, the deformation gradient is a measure of a homogeneous deformation of a body.

Deformation gradient is a linear map. Let us study the linear algebra of the material segment and deformation gradient. When a body undergoes a homogeneous deformation, the material segments in the body in the reference state form one vector space, and the material segments in the body in the current state form another vector space.

Recall the defining attributes of a linear map $\mathbf{F}$ that maps one vector space to another vector space:

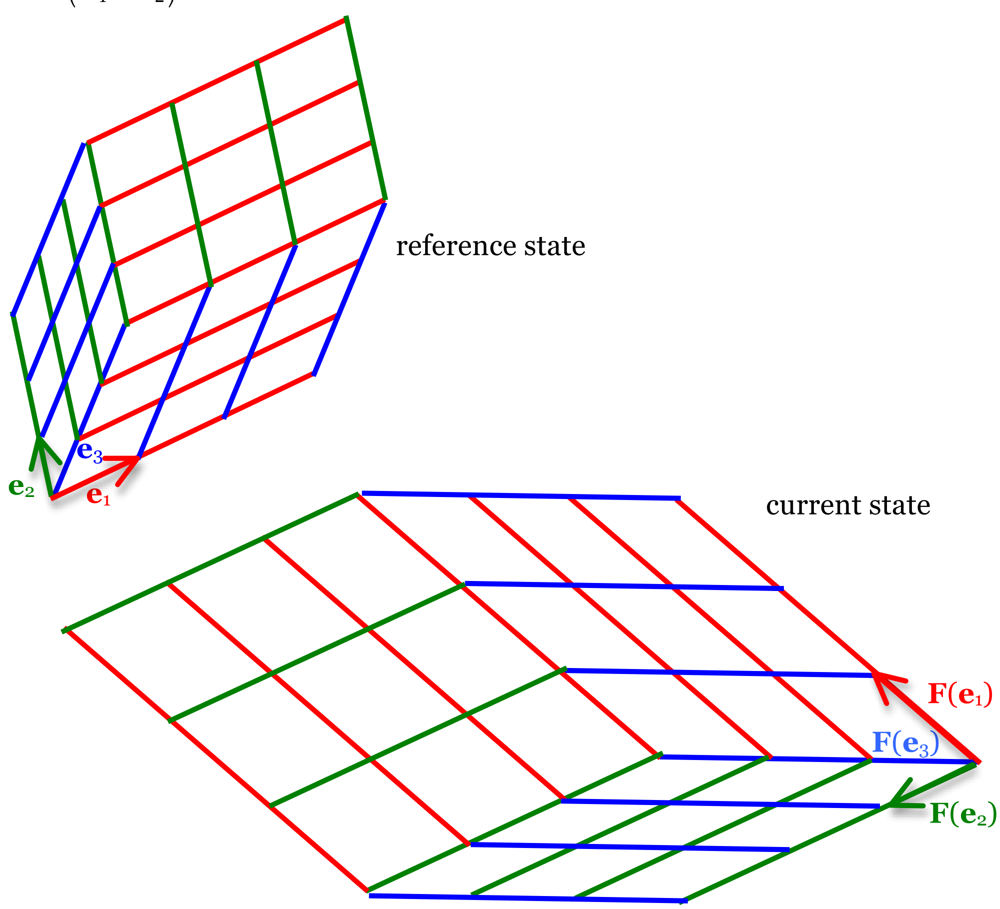
(i) $\mathbf{F}(\alpha\mathbf{Y}) = \alpha\mathbf{F}(\mathbf{Y})$ for every scalar α and every vector $\mathbf{Y}$.

(ii) $\mathbf{F}(\mathbf{Y}_1 + \mathbf{Y}_2) = \mathbf{F}(\mathbf{Y}_1) + \mathbf{F}(\mathbf{Y}_2)$ for any two vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

These attributes are characteristic of a homogeneous deformation.

We interpret attribute (i) as follows. In the reference state, $\mathbf{Y}$ is a set of material particles that forms a straight segment, and $\alpha\mathbf{Y}$ is another set of material particles that also forms a straight segment. The two segments are parallel, and the length of the segment $\alpha\mathbf{Y}$ is α times that of the segment $\mathbf{Y}$. In the current state, the two sets of material particles still form two straight segments: one is $\mathbf{y} = \mathbf{F}(\mathbf{Y})$, and the other is in the same direction as $\mathbf{y}$, and is α times long.

We interpret attribute (ii) as follows. In the reference state, the three vectors $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_1 + \mathbf{Y}_2$ are three sets of material particles that form a triangle. In the current states, the same three sets of material particles form another triangle. The sides of the triangle are the three vectors $\mathbf{F}(\mathbf{Y}_1)$, $\mathbf{F}(\mathbf{Y}_2)$ and $\mathbf{F}(\mathbf{Y}_1 + \mathbf{Y}_2)$.



Components of a vector (i.e., a material segment). A vector like $\mathbf{Y}$ represents a physical object, in this case a material segment in the reference state. All such vectors form a three-dimensional vector space. We can choose three linearly independent vectors in the vector space as a basis. Any vector in the space is a linear combination of the three base vectors. The object—the vector—is independent of the choice of the basis. The components of the vectors, however, do depend on the choice of the basis. When a new basis is chosen, the components of the vectors transform. You have studied rules of transformation in linear algebra.

When the body is in the reference state, a set of material particles forms a parallelepiped, with the three edges represented by vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. When the body is in the current state, the same set of material particles forms another parallelepiped, with three edges represented by vectors $\mathbf{F}(\mathbf{e}_1)$, $\mathbf{F}(\mathbf{e}_2)$ and $\mathbf{F}(\mathbf{e}_3)$.

When the body is in the reference state, consider a particular material particle. Using this material particle as the starting point we can form many straight segments of material particles. Each segment is a vector, and all such segments form a vector space. Let three segments $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be a basis of the vector space. Any segment $\mathbf{Y}$ is a linear combination of the three base vectors:

$$\mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3 .$$

The three quantities Y_1, Y_2 and Y_3 are the components of the vector $\mathbf{Y}$ relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. This algebraic formula has a familiar geometric interpretation. The segment $\mathbf{Y}$ is a diagonal of a parallelepiped, while the vectors $Y_1\mathbf{e}_1$, $Y_2\mathbf{e}_2$ and $Y_3\mathbf{e}_3$ are the edges of the parallelepiped.

We can write the above equation in shorthand:

$$\mathbf{Y} = Y_K\mathbf{e}_K .$$

This way of writing follows a convention: summation is implied over the repeated index.

In the current state, the vector $\mathbf{y}$ is also a linear combination of the three base vectors:

$$\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3 .$$

The three quantities y_1, y_2 and y_3 are the components of the vector $\mathbf{y}$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Following the summation convention, we write the above equation as

$$\mathbf{y} = y_i \mathbf{e}_i .$$

In the above, we have used the same basis for both the reference state and the current state. This practice will force us to abandon a convention we have adopted. While the uppercase **Y** and the lower case **y** represent the same set of material particles in the reference and the current states, the vector $Y_1\mathbf{e}_1$ and $y_1\mathbf{e}_1$ no longer represent the same set of material particles. The homogeneous deformation of the body transforms a parallelepiped in the reference state into another parallelepiped in the current state. Indeed, $Y_1\mathbf{e}_1$ represents the set of material particles that forms one edge of the parallelepiped when the body is in the reference state. The deformation **F** will map this set of material particles to $Y_1\mathbf{F}(\mathbf{e}_1)$, which is in general different from $y_1\mathbf{e}_1$.

Components of the deformation gradient. Similarly, the deformation gradient **F** represents a deformation, and is independent of the choice of the basis. The components of the deformation gradient do depend on the choice of the basis. When a new basis is chosen, the components of the deformation gradient transform. Vectors and linear operators are examples of a more general mathematical object: tensor.

Consider a set of material particles. In the reference state, the set of material particles lies on the base vector $\mathbf{e}_1$. In the current state, the same set of material particles still lies on a straight segment, but the deformation causes the segment to stretch and rotate. By the definition of the deformation gradient **F**, in the current state, the set of material particles forms a segment represented by a vector $\mathbf{F}(\mathbf{e}_1)$. We also use $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ as the basis for the vector space of the material segments in the current state. The vector $\mathbf{F}(\mathbf{e}_1)$ is also a linear combination of the three base vectors. Write

$$\mathbf{F}(\mathbf{e}_1) = F_{11}\mathbf{e}_1 + F_{21}\mathbf{e}_2 + F_{31}\mathbf{e}_3 .$$

The three quantities F_{11}, F_{21} and F_{31} are the components of the vector $\mathbf{F}(\mathbf{e}_1)$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Similarly, we can also consider the deformation of the material particles that lie on the base vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. We write

$$\mathbf{F}(\mathbf{e}_2) = F_{12}\mathbf{e}_1 + F_{22}\mathbf{e}_2 + F_{32}\mathbf{e}_3 ,$$

$$\mathbf{F}(\mathbf{e}_3) = F_{13}\mathbf{e}_1 + F_{23}\mathbf{e}_2 + F_{33}\mathbf{e}_3 .$$

Following the summation convention, we write the above three equations as

$$\mathbf{F}\left(\mathbf{e}_K\right) = F_{iK}\mathbf{e}_i .$$

The nine quantities F_{iK} are the components of the deformation gradient relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. To remind us of the distinct roles played by the two indices, we write the first index in lowercase, and the second index in uppercase. The nine components of the deformation gradient can be listed in a matrix:

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

As a convention, the first index indicates the row, and the second the column. The first column of the matrix, F_{i1}, are the three components of an edge vector of the parallelepiped, $\mathbf{F}\left(\mathbf{e}_1\right)$. Similarly, the second and third columns of the matrix are the components of the other two edge vectors of the parallelepiped. In general, the matrix of the deformation gradient is not symmetric.

Write linear map using components. Write the linear map as

$$\begin{aligned} \mathbf{y} &= \mathbf{FY} \\ &= \mathbf{F}\left(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3\right) \\ &= Y_1\mathbf{F}\left(\mathbf{e}_1\right) + Y_2\mathbf{F}\left(\mathbf{e}_2\right) + Y_3\mathbf{F}\left(\mathbf{e}_3\right) \end{aligned}$$

We then write $\mathbf{F}\left(\mathbf{e}_1\right)$, $\mathbf{F}\left(\mathbf{e}_2\right)$ and $\mathbf{F}\left(\mathbf{e}_3\right)$ as linear combinations of the base vectors, so that

$$\begin{aligned} \mathbf{y} = &\left(F_{11}Y_1 + F_{12}Y_2 + F_{13}Y_3\right)\mathbf{e}_1 \\ &+\left(F_{21}Y_1 + F_{22}Y_2 + F_{23}Y_3\right)\mathbf{e}_2 \\ &+\left(F_{31}Y_1 + F_{32}Y_2 + F_{33}Y_3\right)\mathbf{e}_3 \end{aligned}$$

Consequently, the three components of the vector $\mathbf{y}$ are

$$\begin{aligned} y_1 &= F_{11}Y_1 + F_{12}Y_2 + F_{13}Y_3 \\ y_2 &= F_{21}Y_1 + F_{22}Y_2 + F_{23}Y_3 \\ y_3 &= F_{31}Y_1 + F_{32}Y_2 + F_{33}Y_3 \end{aligned}$$

Using the summation convention, we write the above three equations as

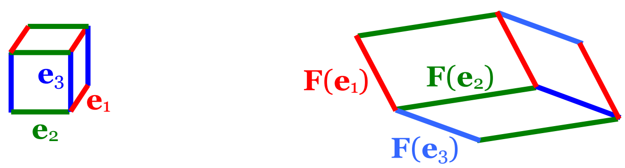
$$y_i = F_{iK}Y_K .$$

Using these components, we write the linear relation $\mathbf{y} = \mathbf{FY}$ as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

We have written the linear map in several equivalent ways: in boldfaced letters $\mathbf{y} = \mathbf{FY}$, in longhand, in shorthand and using a matrix.

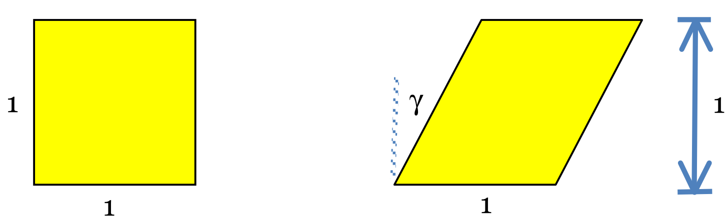
Inner-product space. So far we have only used the property of a vector space without invoking length and angle. Our space, however, does equip with the inner product. So we can speak of the distance between two material particles, and the angle between two lines of material particles.



reference state current state

When the body is in the reference state, we choose a set of material particles that forms a unit cube. The three edges of the unit cube form an orthonormal basis, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. When the body is in the current state, the same set of material particles forms a parallelepiped. The deformation gradient $\mathbf{F}$ maps the three base vectors to the three edges of the parallelepiped, $\mathbf{F}(\mathbf{e}_1)$, $\mathbf{F}(\mathbf{e}_2)$ and $\mathbf{F}(\mathbf{e}_3)$.

Exercise. A body undergoes a shear deformation. Mark a set of material particles in the body. When the body is in the reference state, the set of material particles forms a unit cube. When the body is in the current state, the same set of material particle forms a parallelepiped, as shown in the figure. The dimension normal to the paper (not shown) remains unchanged. Write the deformation gradient for this deformation.



reference state current state

Exercise. A rectangular block deforms into another rectangular block. In the reference state, the lengths of the edges of the rectangular block are of lengths L_1, L_2 and L_3. In the current state, the corresponding edges of the rectangular block are of lengths l_1, l_2 and l_3. The rectangular block in the two states has the same orientation. Write the deformation gradients for the following situations.

(a) The two blocks have the same orientation.

(b) The block in the current state is rotated 90 degrees around the axis of l_3.

(c) The block in the current state is rotated 30 degrees around the axis of l_3.

Exercise. In the reference state, a rectangular block of a material has edges of lengths 1, 2 and 3. We set coordinates in the direction of the three edges of the block. The block undergoes a homogeneous deformation and becomes a parallelepiped. The homogeneous deformation is characterized by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Calculate the vectors formed by the three edges of the parallelepiped. Calculate the lengths of, and angles between, the three edges of the parallelepiped.

Rigid-body rotation does not affect the state of matter. **Green deformation tensor**. The unit cube deforms into the parallelepiped by changing shape, size, and orientation. Only the shape and size of the parallelepiped affect the state of matter. Once the shape and the size of the parallelepiped is fixed, the state of matter is fixed, and is unaffected by any rigid-body rotation of the parallelepiped.

The shape and the size of the parallelepiped are fully specified by the lengths of, and the angles between, the three edges of the parallelepiped. The three lengths and the three angles by themselves do not form a tensor. The six quantities, however, are related to the six inner products of the three edge vectors. The inner product of two edge vectors $\mathbf{F}(\mathbf{e}_K)$ and $\mathbf{F}(\mathbf{e}_L)$ of the parallelepiped is designated as C_{KL}:

$$C_{KL} = \left(\mathbf{F}(\mathbf{e}_K)\right) \cdot \left(\mathbf{F}(\mathbf{e}_L)\right).$$

The six quantities C_{KL} together form a tensor, known as the *Green deformation tensor*. This tensor is positive-definite and symmetric.

The vector $\mathbf{F}(\mathbf{e}_K)$ has three components F_{1K}, F_{2K}, F_{3K}. The vector $\mathbf{F}(\mathbf{e}_L)$ has three components F_{1L}, F_{2L}, F_{3L}. The inner product of the two vectors is

$$C_{KL} = F_{1K}F_{1L} + F_{2K}F_{2L} + F_{3K}F_{3L}.$$

Using the summation convention, we write this inner products as

$$C_{KL} = F_{iK}F_{iL}.$$

The above expression is also written as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}.$$

The components of the tensor form a matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

This matrix is symmetric. Each diagonal element of the matrix is the square of the length of an edge of the parallelepiped, and each off-diagonal element of the matrix is related to the angle between two edges of the parallelepiped.

Both **F** and **C** measure a homogeneous deformation of a body. The deformation gradient **F** includes both the rotation and the distortion of the body. By contrast, the Green deformation tensor **C** measures the distortion only, and is unchanged when the body undergoes any rotation.

Exercise. Show that the Green deformation tensor is symmetric and positive-definite.

Exercise. A body undergoes a homogeneous deformation specified by a deformation gradient. This deformation maps material particles in a unit cube in the reference state to a parallelepiped in the current state. The shape of the parallelepiped depends on the orientation of the unit cube. Let the lengths of the three edges of the parallelepiped be l_1, l_2 and l_3. Show that the sum $l_1^2 + l_2^2 + l_3^3$ is independent of the orientation of the unit cube.

ADDITIONAL RESULTS OF HOMOGENEOUS DEFORMATION

A line of material particles. A body undergoes a homogeneous deformation $\mathbf{F}$ from a reference state to a current state. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms a segment of a straight line, of length L and in the direction of unit vector $\mathbf{M}$. When the body is in the current state, the set of material particles remains as a segment of a straight line, but the segment stretches and rotates: the segment is of length l and in the direction of unit vector $\mathbf{m}$.

The deformation gradient $\mathbf{F}$ maps the segment in the reference state to the segment in the current state:

$$l\mathbf{m} = \mathbf{F}\left(L\mathbf{M}\right).$$

Recall the definition of the stretch of the line of material particles, $\lambda = l / L$, and we write the above equation as

$$\lambda\mathbf{m} = \mathbf{FM}.$$

Given a deformation gradient $\mathbf{F}$, and given the direction $\mathbf{M}$ of a line of material particles in the reference state, this equation calculates the stretch λ and the direction $\mathbf{m}$ of the line of material particles in the current state.

Exercise. Show that

$$\lambda^2 = \left(\mathbf{FM}\right)^T\left(\mathbf{FM}\right) = \mathbf{M}^T\mathbf{CM}.$$

Exercise. The inner product $\mathbf{M}\cdot\mathbf{m}$ gives the cosine of the angle between the two vectors—that is, the inner product tells us how much the deformation of the body rotates the line of material particles. Show that this inner product is given by

$$\mathbf{M}\cdot\mathbf{m} = \frac{\mathbf{M}^T\mathbf{FM}}{\lambda}.$$

Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Consider a set of material particles that forms a segment of a straight line. When the body is in the reference state, the material particles at the two ends of the segment are at two points (0,0,0) and (2,3,6). Calculate the stretch and the direction of the segment when the body is in the current state.

Two lines of material particles. Recall the definition of the engineering shear strain. Consider two sets of material particles. In the reference state, the two sets of material particles form two orthogonal vectors. In the current state, the two sets of material particles still form two vectors, but they are, in general, no longer orthogonal to each other. Let the angle between the two lines of material particles in the current state be $\frac{\pi}{2}-\gamma$. The angle γ is defined as the shear strain.

reference state current state

Consider two sets of material particles. In the reference state, one set of material particles forms a unit vector $\mathbf{M}$, and the other set of material particles forms a unit vector $\mathbf{N}$; the two vectors are orthogonal. After the body undergoes a homogeneous deformation $\mathbf{F}$, the two sets of material particles stretch by $\lambda_{\mathbf{M}}$ and $\lambda_{\mathbf{N}}$, and are in the directions of two vectors $\mathbf{m}$ and $\mathbf{n}$. Each set of material particles is linearly mapped by the deformation:

$$\lambda_{\mathrm{M}}\mathbf{m} = \mathbf{FM},$$

$$\lambda_{\mathrm{N}}\mathbf{n} = \mathbf{FN}.$$

Taking inner products of the vectors, we obtain that

$$\lambda_{\mathrm{M}}\lambda_{\mathrm{N}}\mathbf{m}^T\mathbf{n} = (\mathbf{FM})^T(\mathbf{FN}).$$

On the left-hand side, the inner product gives the cosine of the angle between the two vectors, $\mathbf{m}^T\mathbf{n} = \cos\left(\frac{\pi}{2}-\gamma\right) = \sin\gamma$. Consequently, the above expression can be written as

$$\sin\gamma = \frac{(\mathbf{FM})^T(\mathbf{FN})}{\lambda_{\mathbf{M}}\lambda_{\mathbf{N}}}.$$

Given the directions of two orthogonal lines of material particles in the reference state, **M** and **N**, and given the deformation gradient **F**, the above equation calculates the shear strain associated with the two lines of material particles in the current state.

Exercise. Show that

$$\sin\gamma = \frac{\mathbf{M}^T\mathbf{C}\mathbf{N}}{\lambda_{\mathbf{M}}\lambda_{\mathbf{N}}}.$$

This result confirms that the angle between two lines of material particles do not change when the body undergoes a rigid-body rotation.

Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Consider two sets of material particles. When the body is in the reference state, one set of material particles forms a straight line passing through two points (0,0,0) and (2, 3, 6), and the other set of material particles forms a straight line passing through two points (0,0,0) and (-6, -2, 3). Calculate the angle between the two lines of material particles when the body is in the current state.

A volume of material particles. Once again consider a homogeneous deformation **F** that changes a unit cube to a parallelepiped. Let the three edges of the parallelepiped be the vectors **a**, **b** and **c**. The volume of the parallelepiped is $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Recall that the three vectors are the columns of the matrix **F**. Further recall that $\det\mathbf{F} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Consequently, the volume of the parallelepiped is $\det\mathbf{F}$.

We next generalize the above result to a homogeneous deformation of a body of any shape. The body occupies a region of volume of V in the reference state, and occupies another region of volume v in the current state. The two volumes are related as

$$v = V\det\mathbf{F}.$$

Thus, the deformation gradient is expected to obey $\det\mathbf{F} > 0$. The ratio of the volumes in the two states is often written in shorthand as $J = \det\mathbf{F}$.

Inverse map. A linear map **F** is singular when $\det\mathbf{F} = 0$. A theorem in linear algebra says that, if and only if the map is nonsingular, $\det\mathbf{F} \neq 0$, the linear map can be inverted. Write the inverse of **F** as $\mathbf{F}^{-1}$. Consider a set of material particles in the body. When the body is in the current state, the set of material particles forms a vector $\mathbf{y}$. When the body is in the reference state, the set of material particles forms a vector $\mathbf{Y}$. The two vectors relate to each other by the maps:

$$\mathbf{y} = \mathbf{F}\mathbf{Y},$$

$$\mathbf{Y} = \mathbf{F}^{-1}(\mathbf{y}).$$

In general, we require that the deformation gradient be nonsingular.

A plane of material particles. Consider a set of material particles. In the reference state, the set of material particles lies in a region in a plane, unit normal $\mathbf{N}$ and area A. In the current state, the same set of martial particles is lies in a region in another plane, unit normal $\mathbf{n}$ and area a. The deformation maps the same set of material particles from one region in the reference state to the other region in the current state.

The body may undergo a shear deformation relative to the plane. Consider a line of material particles. When the body is in the reference state, the line of material particles is normal to the plane of material particle. When the body is in the current state, the line of material particles may no longer be normal to the plane of material particles. Consequently, when the body shears relative to the plane, the two normal directions $\mathbf{N}$ and $\mathbf{n}$ consist of two distinct sets of material particles. The use of the uppercase and lowercase of the same letter is an exception to our convention.

We want to relate $\mathbf{n}$ and a to $\mathbf{N}$ and A. Define area vectors $\mathbf{a} = a\mathbf{n}$ and $\mathbf{A} = A\mathbf{N}$. Consider a tilted cone with the plane as the base, and an arbitrary point in space as the apex. Pick a material particle in the plane. Consider the segment of the line of material particles from the apex to the point. Denote the segment by vector $\mathbf{Y}$ when the body is in the reference state, and by $\mathbf{y}$ when the body is in the current state. The volume of the cone is $V = \mathbf{Y} \cdot \mathbf{A}/3$ in the reference state, and is $v = \mathbf{y} \cdot \mathbf{a}/3$ in the current state. Recall that $\mathbf{y} = \mathbf{F}\mathbf{Y}$ and $v = JV$, so that

$$\mathbf{Y}^T\mathbf{F}^T\mathbf{a} = J\mathbf{Y}^T\mathbf{A}$$

This expression equates two scalars. Each scalar is an inner product of one vector and another vector $\mathbf{Y}$. Because the equation holds for any arbitrary pick of the material particle in the base of the cone, the equation must hold for arbitrary vector $\mathbf{Y}$. Consequently, the above equation implies an equation between two vectors:

$$\mathbf{F}^T\mathbf{a} = J\mathbf{A}.$$

This relation is known as the *formula of Nanson*. In terms of components, this formula is

$$an_iF_{iK} = JAN_K.$$

When the deformation gradient is known, this equation can be used to calculate $\mathbf{n}$ and a in terms of $\mathbf{N}$ and A.

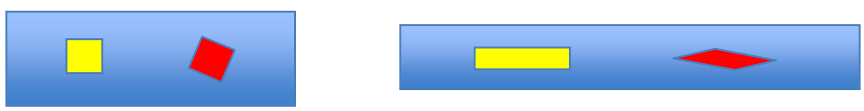
Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

In the reference state, a set of material particles lies on a plane, within a region of unit area; the unit vector normal to the plane is (2/7, 3/7, 6/7). The homogeneous deformation moves the same set of material particles to places that lie within a region of some other shape, on a plane of some other direction. Calculate the area of the region and the unit vector normal to the plane in the current state.

Principal directions of a deformation. When a rod is pulled, it elongates in the axial direction and contracts in the two transverse directions. Consider a set of material particles in the rod. When the rod is in the reference state, the set of material particles form a unit cube with edges in the axial and transverse directions of the rod. When the rod is in the current state, the same set of material particles forms a rectangular block.

Now consider another set of material particles. When the rod is in the reference state, the set of material particles forms a unit cube with edges *not* in the axial and transverse directions of the rod. When the rod is in the current state, the same set of material particles forms a parallelepiped.



reference state current state

As a further example, consider a body undergoing a shear deformation. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms a unit cube. When the body is in the current state, the same set of material particles forms a parallelepiped.

Now consider another set of material particles. When the body is in the reference state, the set of material particles forms a unit cube oriented in a particular direction, such that, when the body is in the current state, the same set of material particles forms a rectangular block.

reference state current state

We now generalize these observations to a body undergoing an arbitrary homogeneous deformation **F**. Consider a set of material particles. When the body is in the reference state, the set of material particles forms a unit cube. When the body is in the current state, the same set of material particles forms a parallelepiped. The shape of the parallelepiped in the current state depends on the orientation of the unit cube in the reference state.

For a particular choice of the orientation of the unit cube, however, the deformed shape in the current state can be a rectangular block. The deformation stretches the unit cube into a rectangular block, and also rotates the rectangular block. The directions of the three edges of the unit cube in the reference state are called the principal directions of the deformation. The lengths of the three edges of the rectangular block in the current state are called the principal stretches. We next use the deformation gradient **F** to calculate the principal directions and principle stretches, as well as the rotation between the unit cube and rectangular block.

Exercise. According to a theorem in linear algebra, a symmetric operator has three real eigenvalues. Furthermore, if the eigenvalues are distinct, the corresponding eigenvectors are orthogonal to one another. Prove this theorem. What happens if the three eigenvalues are not distinct?

Exercise. A body undergoes a homogeneous deformation **F** from a reference state to a current state. The deformation gradient **F** maps a straight line of material particles in the reference state to a straight line in the current state. As the body deforms, the line of material particles rotate and stretch. The stretch of the line of material particles depends on the orientation of the line in the reference state. Determine the orientation of the straight line that maximize or minimize the stretch.

Eigenvectors of the Green deformation tensor. The Green deformation tensor **C** is symmetric and positive-definite. According to a theorem in linear algebra, a symmetric and positive-definite matrix has three orthogonal eigenvectors, along with three real and positive eigenvalues.

Let $\mathbf{M}$ be the unit vector in the direction of an eigenvector of **C**, and α be the associated eigenvalue. According to the definition of the eigenvalue and eigenvectors, we write

$$\mathbf{CM} = \alpha \mathbf{M} .$$

The deformation causes the line of material particles to stretch by λ, which is calculated from

$$\lambda^2 = \mathbf{M}^T \mathbf{CM} .$$

A comparison of the above two expressions gives that

$$\lambda^2 = \alpha .$$

The eigenvalue of **C** is the principal stretch squared.

Consider two lines of material particles. When the body is in the reference state, the two lines are orthogonal. When the body is in the current state, in general, the two lines are not orthogonal. The change in the angle between the two lines measures the shear. However, if in the reference state the two lines of material particles are in the directions of two orthogonal eigenvectors, in the current state the two lines of material particles will remain orthogonal. This property can be verified as follows. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the unit vectors in the directions of two orthogonal eigenvectors of **C**. Let λ_1 and λ_2 be the two principal stretches. In the current state, the two lines of material particles are in the directions of unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$. We know that

$$\lambda_1 \mathbf{m}_1 = \mathbf{FM}_1 ,$$

$$\lambda_2 \mathbf{m}_2 = \mathbf{FM}_2 .$$

The inner product of the two vectors gives that

$$\lambda_1 \lambda_2 \mathbf{m}_1 \cdot \mathbf{m}_2 = \mathbf{M}_1 \cdot \left(\mathbf{CM}_2 \right) .$$

Because $\mathbf{M}_2$ is an eigenvector, $\mathbf{CM}_2 = \lambda_2^2 \mathbf{M}_2$. Recall that $\mathbf{M}_1$ and $\mathbf{M}_2$ are orthogonal, $\mathbf{M}_1 \cdot \mathbf{M}_2 = 0$, so that $\mathbf{m}_1 \cdot \mathbf{m}_2 = 0$. The deformation rotates both lines of material particles, but keeps the two lines orthogonal to each other.

Here is how we choose a particular set of material particles. When the body is in the reference state, the set of material particles form a unit cube whose edges are in the directions of three orthogonal eigenvectors of the deformation tensor **C**. When the body is in the current state, the same set of material particles forms a rectangular block, with the three edges of the length of the principal stretches. The rectangular block may be rotated from the unit cube.

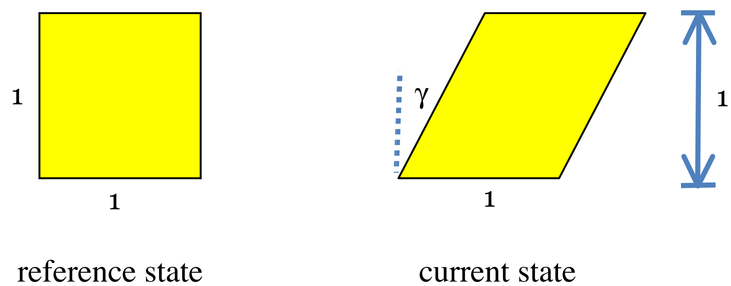
Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Calculate the principal directions of the deformation in the reference state
- Calculate the principal stretches.

Calculate the principal directions of the deformation in the current state.

Exercise. A body undergoes a shear deformation. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles is a unit cube. When the body is in the current state, the same set of material particle is a parallelepiped, as shown in the figure. The dimension normal to the paper (not shown) remains unchanged. Calculate the principal directions and principal stretches.



Represent a symmetric operator in terms of its eigenvectors and eigenvalues. Recall a procedure in linear algebra. Let the orthonormal eigenvectors of a symmetric operator $\mathbf{C}$ be $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ and the corresponding eigenvalues be $\alpha_1, \alpha_2, \alpha_3$. Write

$$\mathbf{G} = \left[\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\right],$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}.$$

The operator $\mathbf{C}$ can be represented by

$$\mathbf{C} = \mathbf{G}\tilde{\mathbf{C}}\mathbf{G}^T.$$

We can verify this representation as follows. Let $\mathbf{Y}$ be a vector. Because

$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ are linearly independent, we can write any vector $\mathbf{Y}$ as a linear combination

$$\mathbf{Y} = Y_1\mathbf{M}_1 + Y_2\mathbf{M}_2 + Y_3\mathbf{M}_3 .$$

Here Y_1, Y_2, Y_3 are components of the vector $\mathbf{Y}$ in the three directions $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$. Note that

$$\mathbf{CY} = Y_1\alpha_1\mathbf{M}_1 + Y_2\alpha_2\mathbf{M}_2 + Y_3\alpha_3\mathbf{M}_3 .$$

Also note that

$$\left(\mathbf{G}\tilde{\mathbf{C}}\mathbf{G}^T\right)\mathbf{Y} = Y_1\alpha_1\mathbf{M}_1 + Y_2\alpha_2\mathbf{M}_2 + Y_3\alpha_3\mathbf{M}_3 .$$

A comparison of the above two expressions confirms that $\mathbf{C} = \mathbf{G}\tilde{\mathbf{C}}\mathbf{G}^T$.

Stretch tensor. The Green deformation tensor $\mathbf{C}$ is a symmetric, positive-definite tensor. Define another symmetric, positive-definite tensor $\mathbf{U}$ by

$$\mathbf{C} = \mathbf{U}^2$$

The tensor $\mathbf{U}$ is called the stretch tensor.

Given $\mathbf{C}$, we can calculate the stretch tensor $\mathbf{U}$ as follows. Let the orthonormal eigenvectors of a symmetric operator $\mathbf{C}$ be $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ and the corresponding eigenvalues be $\alpha_1, \alpha_2, \alpha_3$. Write

$$\mathbf{G} = \left[\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\right],$$

and

$$\tilde{\mathbf{C}} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}.$$

The operator $\mathbf{C}$ can be represented by

$$\mathbf{C} = \mathbf{G}\tilde{\mathbf{C}}\mathbf{G}^T .$$

Because $\mathbf{C}$ is positive-definite, we define three positive roots (the stretches) $\lambda_1 = \sqrt{\alpha_1}, \lambda_2 = \sqrt{\alpha_2}, \lambda_3 = \sqrt{\alpha_3}$. We can form another diagonal matrix:

$$\tilde{\mathbf{U}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

The stretch tensor is given by

$$\mathbf{U} = \mathbf{G}\tilde{\mathbf{U}}\mathbf{G}^T .$$

We can readily confirm that $\mathbf{U}$ is symmetric and positive-definite, and $\mathbf{C} = \mathbf{U}^2$.

Both $\mathbf{C}$ and $\mathbf{U}$ are symmetric and positive-definite. The two operators have the same eigenvectors. Each eigenvalue of $\mathbf{C}$ is a principal stretch squared. Each eigenvalue of $\mathbf{U}$ is a principal stretch.

Polar decomposition. Here is yet another theorem in linear algebra. Let $\mathbf{F}$ be a linear operator. The operator is nonsingular, i.e., $\det\mathbf{F} \neq 0$. The linear operator can be written as a product:

$$\mathbf{F} = \mathbf{RU},$$

where $\mathbf{R}$ is an orthogonal operator, satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, and $\mathbf{U}$ is a symmetric, positive-definite operator. Writing a linear operator in this way is known as polar decomposition.

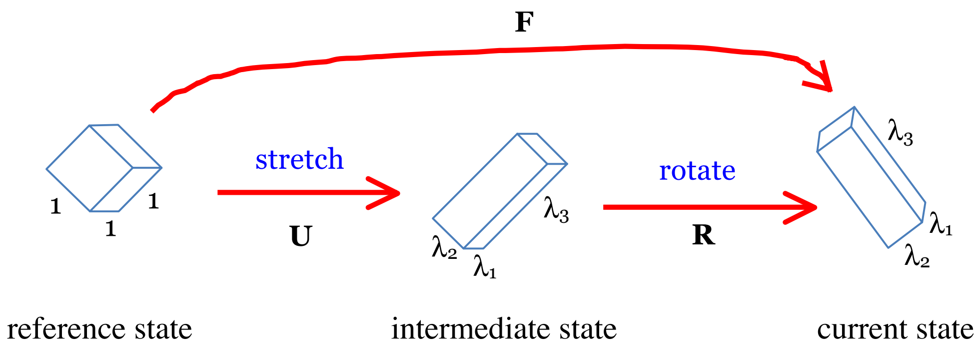
The proof of this theorem is straightforward. Because $\mathbf{F}$ is nonsingular, the product $\mathbf{F}^T\mathbf{F}$ is a symmetric, positive-definite operator. Thus, we can find a symmetric, positive-definite operator $\mathbf{U}$ to satisfy $\mathbf{F}^T\mathbf{F} = \mathbf{U}^2$. Furthermore, we can confirm that $\mathbf{FU}^{-1}$ is an orthogonal operator, namely,

$$\left(\mathbf{FU}^{-1}\right)^T\left(\mathbf{FU}^{-1}\right) = \left(\mathbf{U}^{-1}\mathbf{F}^T\right)\left(\mathbf{FU}^{-1}\right) = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I} .$$

Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\mathbf{F} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Calculate $\mathbf{U}$ and $\mathbf{R}$.



Geometric interpretation of polar decomposition. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms a unit cube, whose three edges are in the directions of the eigenvectors of the deformation tensor **C**. In the current state, the block becomes a rectangular block, whose three edges are of the length of the principal stretches. The rectangular block may be rotated from the unit cube.

The multiplication $\mathbf{F} = \mathbf{RU}$ means two linear maps in succession. For example, start with a set of material particles that forms a unit cube in the reference state, with the edges of the cube oriented in the directions of the eigenvectors of **U**.

The operator **U** stretches in the unit cube into a rectangular block in an intermediate state. Because the three edges of the unit cube are in the directions of the eigenvectors of **U**, the rectangular block in the intermediate state does not rotate relative to the unit cube in the reference state.

The operator **R** then rotates the rectangular block in the intermediate state to the rectangular block in the current state. Because **R** is a rotation operator, the rectangular block is rotated as a rigid body, with no stretch.

Exercise. A body undergoes a homogeneous deformation **F** from a reference state to a current state. Mark a set of material particles in the body. When the body is in the reference state, the set of material particles lies on the surface of a unit sphere. What do the operators **U**, **R** and **F** do to this set of material particles?

Exercise. Show that a non-singular linear operator **F** can be written as

$$\mathbf{F} = \mathbf{VR},$$

where **R** is an orthogonal operator, and **V** a symmetric operator. When **F** is the deformation gradient, interpret the roles of **V** and **R** in geometric terms. **V** is known as the left stretch tensor.

Displacement gradient. When a body undergoes a homogeneous deformation, a material particle in the body moves from position **X** in the reference state to position **x** in the current state. The displacement of the material particle is

$$\mathbf{u} = \mathbf{x} - \mathbf{X}.$$

If the body undergoes a rigid-body translation, all material particles in the body move by the same displacement. If the body also rotates and stretches, however, different material particles in the body can move by different displacements. Consider another material particle, whose position is $\mathbf{X}_o$ in the reference state and is $\mathbf{x}_o$ in the current state. The displacement of this material particle is

$$\mathbf{u}_o = \mathbf{x}_o - \mathbf{X}_o .$$

Define a new tensor $\mathbf{H}$ by

$$\mathbf{u} - \mathbf{u}_o = \mathbf{H}\left(\mathbf{X} - \mathbf{X}_o\right)$$

The tensor $\mathbf{H}$ is called the *displacement gradient*. Note that $\mathbf{Y} = \mathbf{X} - \mathbf{X}_o$ and $\mathbf{y} = \mathbf{x} - \mathbf{x}_o$. A comparison with $\mathbf{y} = \mathbf{F}\left(\mathbf{Y}\right)$ gives that

$$\mathbf{H} = \mathbf{F} - \mathbf{I} .$$

Small-strain approximation. Recall that $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. We can express the Green deformation tensor in terms of the displacement gradient:

$$\mathbf{C} = \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H} + \mathbf{I} .$$

When all the components of the displacement gradient is small, $H_{iK} << 1$, we can neglect the quadratic term, and write

$$\mathbf{C} \approx \mathbf{H} + \mathbf{H}^T + \mathbf{I} .$$

Recall that in the small-strain approximation, the strain relates to the displacement gradient as

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right).$$

Except for the factor 2 and the identity tensor, the above two expressions coincide. In finite deformation, the Green deformation tensor $\mathbf{C}$ is a measure of deformation unaffected by rigid-body rotation. The small-strain approximation is valid when all components of the displacement gradient are small.

Lagrange strain. A quantity slightly different from the Green deformation tensor $\mathbf{C}$ is defined by

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right),$$

where $\mathbf{I}$ is the identity tensor. The tensor $\mathbf{E}$ is called the Lagrange strain.

We can relate the general definition of the Lagrange strain to that introduced in describing a tensile bar (http://imechanica.org/node/5065). When the length of the bar increases from L to l, the stretch is defined by $\lambda = l / L$, and the Lagrange strain is defined as

$$\eta = \frac{1}{2}\left(\lambda^2 - 1\right).$$

It was hard to motivate this definition in one dimension. In three dimensions, this definition can be motivated as follows.

A body undergoes a homogeneous deformation specified by the deformation gradient **F**. Consider a set of material particles. In the reference state, the set of material particles forms a segment of a straight line, length L and direction **M**. In the current state, the same set of material particles still form a segment of a straight line, length l and direction **m**. The two segments are related by $l\mathbf{m} = L\mathbf{FM}$. The inner product of the vector gives $l^2 = L^2\mathbf{M}^T\mathbf{CM}$. The change in the length of the segments can be calculated from

$$l^2 - L^2 = 2L^2\mathbf{M}^T\mathbf{EM}$$

The homogeneous deformation causes the line of material particles to stretch by $\lambda = l / L$. Consequently, the Lagrange strain of the element in direction **M** is given by

$$\eta = \mathbf{M}^T\mathbf{EM}.$$

Thus, once we know the tensor **E**, we can calculate the Lagrange strain η of a line of material particles in any direction **M**.

The Lagrange strain relates to the displacement gradient as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\right).$$

When all the components of the displacement gradient is small, $H_{iK} << 1$, we can neglect the quadratic term, and the Lagrange strain coincides with the small-strain approximation:

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right).$$

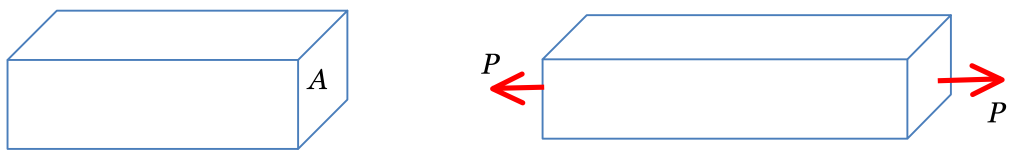
STRESS

Nominal stress. Subject to an axial force, the rod changes its length and cross-sectional area. The nominal stress is defined by

$$\text{nominal stress} = \frac{\text{force in current state}}{\text{area in reference state}}.$$

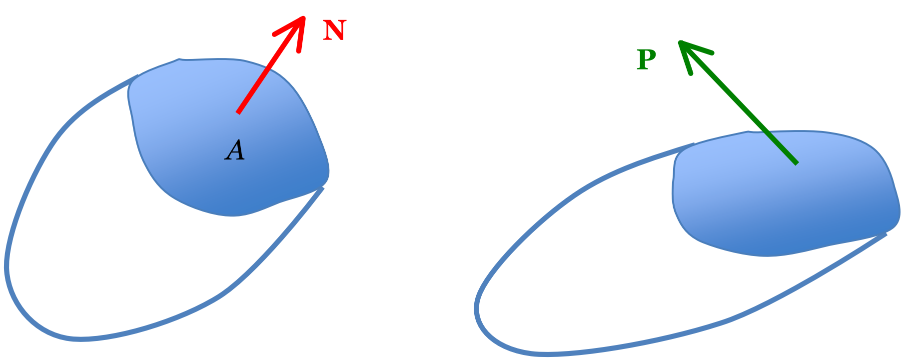
Let A be the cross-sectional area of the rod in the reference state, which is subject to no force. Let P be the axial force in the current state. The nominal stress is defined by

$$s = \frac{P}{A}.$$



reference state current state

We now generalize this definition to a body of an arbitrary shape undergoing a homogeneous deformation of an arbitrary type. Consider a set of material particles. When the body is in the reference state, the set of material particles lies on a plane of unit vector $\mathbf{N}$, in a region of area A. The vector $A\mathbf{N}$ represents the planar region as a vector, written as $\mathbf{A} = A\mathbf{N}$. When the body is in the current state, the same set of material particles forms another planar region, and acting on the planar region is a force $\mathbf{P}$.



reference state current state

Define the nominal stress as an operator $\mathbf{s}$ that maps the area vector $\mathbf{A}$ in the reference state to the force $\mathbf{P}$ in the current state:

$$\mathbf{P} = \mathbf{s}(\mathbf{A}).$$

The shape of the region does not affect this definition. The nominal stress is also known as the first Piola-Kirchhoff stress.

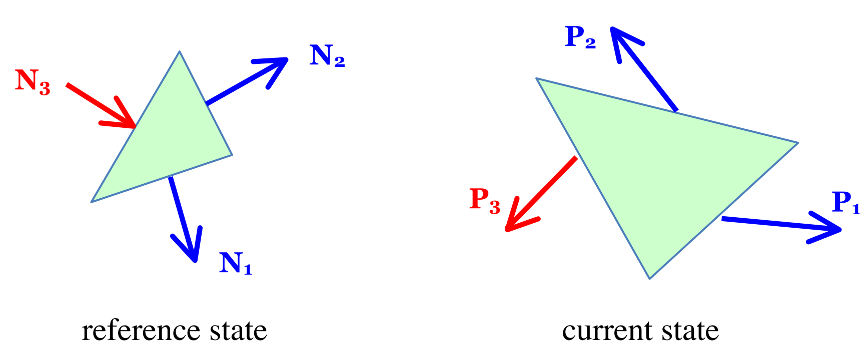
Nominal stress is a linear operator. Let us see why the stress should be a linear operator. If the area vector is scaled by a scalar α, the force should also be scaled, so that

$$\mathbf{s}(\alpha \mathbf{A}) = \alpha \mathbf{s}(\mathbf{A}).$$

Next consider two sets of material particles. When the body is in the reference state, the two sets of material particles form area vectors $\mathbf{A}_1$ and $\mathbf{A}_2$. Because the shapes of the two regions do not affect the definition, we may choose the two regions as rectangular regions. The vector sum $\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$ corresponds to another planar region of material particles. The three regions form the surfaces of a prism. The cross section of the prism is shown in the figure. Note that if the normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ point toward the exterior of the prism, $\mathbf{N}_3$ points toward the interior of the prism. When the body is in the current state, the three sets of material particles deform to some other planar regions, and the forces acting on the three regions are $\mathbf{P}_1 = \mathbf{s}(\mathbf{A}_1)$, $\mathbf{P}_2 = \mathbf{s}(\mathbf{A}_2)$ and $\mathbf{P}_3 = \mathbf{s}(-\mathbf{A}_3)$. The prism is a free-body diagram. The forces acting on the three faces are balanced, $\mathbf{P}_3 + \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{0}$, so that

$$\mathbf{s}(\mathbf{A}_3) = \mathbf{s}(\mathbf{A}_1) + \mathbf{s}(\mathbf{A}_2).$$

We have confirmed that the nominal stress should be a linear operator.



Exercise. A body is subject to a state of nominal stress $\mathbf{s}$ in the current state. Consider a set of material particles. When the body is in the reference state, the set of material particles forms a three-dimensional region bounded by two surfaces. One surface is flat, of area A and unit normal vector $\mathbf{N}$. The other surface is curved. Calculate the total force acting on the curved surface.

Exercise. Give a physical interpretation of the expression

$$\mathbf{s}(-\mathbf{A}) = -\mathbf{s}(\mathbf{A}).$$

Components of nominal stress. The preceding definition is independent the choice of the basis of the vector space. We next choose an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms a unit cube, with the three edges coinciding with the three base vectors. When the body is in the current state, the set of material particles for a parallelepiped.

The deformation maps the face of unit cube normal to $\mathbf{e}_1$ to a face of the parallelepiped. Acting on the face of the parallelepiped is the force $\mathbf{s}(\mathbf{e}_1)$. This force is a vector, which is also a linear combination of the three base vectors:

$$\mathbf{s}(\mathbf{e}_1) = s_{11}\mathbf{e}_1 + s_{21}\mathbf{e}_2 + s_{31}\mathbf{e}_3,$$

where s_{i1} are the three components of the force relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Similarly, we write

$$\mathbf{s}(\mathbf{e}_2) = s_{12}\mathbf{e}_1 + s_{22}\mathbf{e}_2 + s_{32}\mathbf{e}_3,$$

$$\mathbf{s}(\mathbf{e}_3) = s_{13}\mathbf{e}_1 + s_{23}\mathbf{e}_2 + s_{33}\mathbf{e}_3.$$

The nine quantities s_{iK} are the components of the nominal stress. The first index shows the direction of the force in the current state, and the second index shows the direction of the vector normal to the face in the reference state. To remind us of the distinct roles played by the two subscripts, we write the first subscript in lowercase, and the second subscript in uppercase.

Using the summation convention, we write the above three expressions as

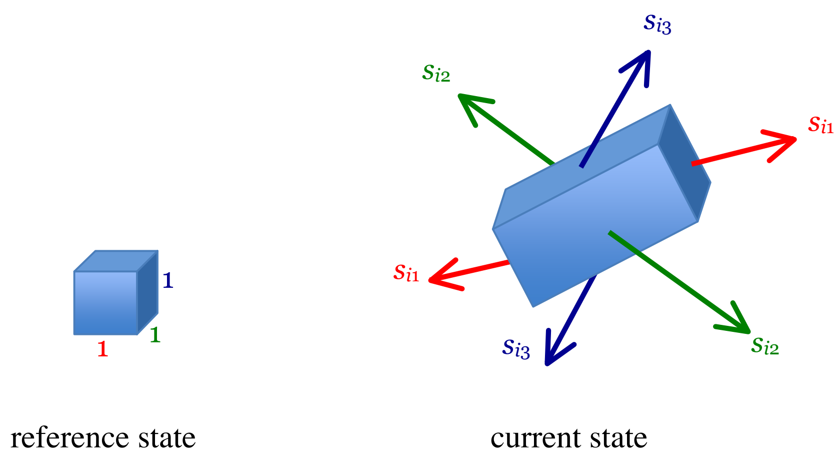
$$\mathbf{s}(\mathbf{e}_K) = s_{iK}\mathbf{e}_i.$$

The nine components of the nominal stress can be listed as a matrix:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

As a convention, the first index indicates the row, and the second the column. In general, the matrix of the nominal stress is not symmetric.

The components of the nominal stress have clear physical significance. The first column of the matrix s_{i1} corresponds to the three components of the force $\mathbf{s}(\mathbf{e}_1)$, acting on the parallelepiped on the face whose normal in the reference state is $\mathbf{e}_1$. The force acting on the parallelepiped on the face whose normal in the reference state is $-\mathbf{e}_1$ is given by $\mathbf{s}(-\mathbf{e}_1) = -\mathbf{s}(\mathbf{e}_1)$. This algebra is consistent with a physical requirement: the balance of the forces acting on the parallelepiped requires that the two forces acting on each pair of parallel faces of the parallelepiped be equal in magnitude and opposite in direction. Similarly, the other two columns of the matrix of nominal stress, s_{i2} and s_{i3}, are the forces acting on the other faces of the parallelepiped.



Stress-traction relation. Consider again the set of material particles. When the body is in the reference state, the set of material particle forms a planar region of unit normal **N** and area A. When the body is in the current state, the set of material particles forms a planar region of some other orientation and area. In the current state, acting on the region is a force **P**. Define the nominal traction **T** by the force acting on the planar region in the current state divided by the area of the region in the reference state:

$$\mathbf{T} = \mathbf{P} / A .$$

Recall the definition of the nominal stress, $\mathbf{P} = \mathbf{s}(\mathbf{A})$. In particular, the area of the region A is a scalar, and the nominal stress is a linear operator, so that $\mathbf{s}(A\mathbf{N}) = A\mathbf{s}(\mathbf{N})$. We obtain that

$$\mathbf{T} = \mathbf{s}(\mathbf{N}).$$

This relation connects the nominal stress and the nominal traction.

The stress-traction relation can be expressed in terms of the components relative to a basis of the vector space, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The normal vector is a linear combination of the base vectors:

$$\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3 .$$

Recall the definition of the components of stress:

$$\mathbf{s}\left(\mathbf{e}_K\right) = s_{1K}\mathbf{e}_1 + s_{2K}\mathbf{e}_2 + s_{3K}\mathbf{e}_3 .$$

Consequently, the linear map of the vector is

$$\begin{aligned}\mathbf{s}\left(\mathbf{N}\right) = &\left(s_{11}N_1 + s_{12}N_2 + s_{13}N_3\right)\mathbf{e}_1 \\ &+\left(s_{21}N_1 + s_{22}N_2 + s_{23}N_3\right)\mathbf{e}_2 \\ &+\left(s_{31}N_1 + s_{32}N_2 + s_{33}N_3\right)\mathbf{e}_3\end{aligned}$$

Recall that $\mathbf{T} = \mathbf{s}\left(\mathbf{N}\right)$. The traction is also a linear combination of the base vectors:

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3 .$$

A comparison of these expressions gives that

$$\begin{aligned}T_1 &= s_{11}N_1 + s_{12}N_2 + s_{13}N_3 \\ T_2 &= s_{21}N_1 + s_{22}N_2 + s_{23}N_3 \\ T_3 &= s_{31}N_1 + s_{32}N_2 + s_{33}N_3\end{aligned}$$

The relation can also be expressed in the matrix form:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

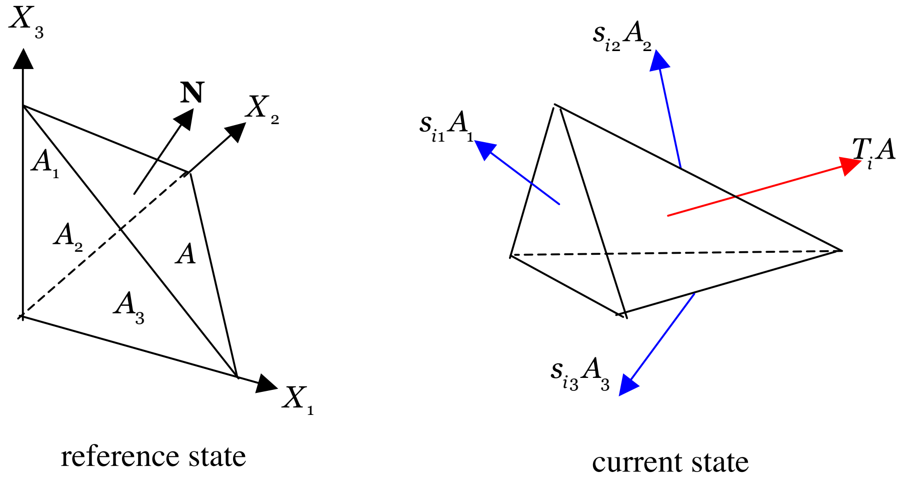
The stress-traction relation is also written in shorthand by adopting the convention of summing over repeated indices:

$$T_i = s_{iK}N_K .$$

Balance of forces. The stress-traction relation can also be derived by balancing forces. Once the state of stress of a material particle is specified by s_{iK}, we know traction on all six faces of the block around the particle. This information is sufficient for us to calculate the force acting on a plane of any direction.

Consider a set of material particles in the body. When the body is in the reference state, the set of material particles forms a tetrahedron, with three faces on the coordinate planes, and one face on the plane normal to the unit vector **N**. Let the areas of the three triangles on the coordinate planes be A_K, and the area of the triangle normal to **N** be A. The geometry dictates that

$$A_K = AN_K .$$



When the body is in the current state, the tetrahedron deforms to another tetrahedron, a part of a parallelepiped. Regard this deformed tetrahedron in the current state as a free-body diagram. On face A_1, the force is $s_{i1}A_1$. On face A_2, the force is $s_{i2}A_2$. On face A_3, the force is $s_{i3}A_3$. On face A, the force is T_iA. That is, T_i is the force acting on the face in the current state divided by the area of the face in the reference state; T_i is known as the normal traction.

Now balance the forces acting on the tetrahedron in the current state. Acting on each of the four faces is a surface force. As the volume of the tetrahedron decreases, the ratio of area over volume becomes large, so that the surface forces prevail over the body force and the inertial force. Consequently, the surface forces on the four faces of the tetrahedron must balance, giving

$$s_{i1}A_1 + s_{i2}A_2 + s_{i3}A_3 = T_iA.$$

This equation, in combination with $A_K = AN_K$, gives

$$s_{i1}N_1 + s_{i2}N_2 + s_{i3}N_3 = T_i.$$

The stress-traction relation is an algebraic expression of a physical law: forces acting on a tetrahedron are balanced.

Exercise. A body undergoes a homogeneous deformation from a reference state to a current state. Consider in the body a set of material particles. When the body is in the reference state, the set of material particles form a tetrahedron, with three faces A_1, A_2, A_3 on the coordinate planes, and the fourth face A intersects the three coordinate axes at 1, 2, and 3. In the current state, the same set of material particles forms a tetrahedron of some other shape, and the forces acting on all faces are in the direction of X_1, with the forces on faces A_1, A_2, A_3 being of magnitudes 4, 5, 6, respectively. Calculate the force on face A. Calculate the nominal stress tensor.

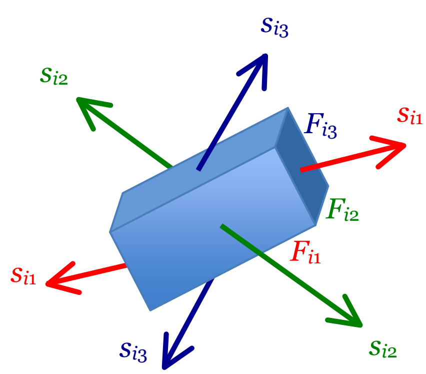
Balance of moments. Let $\mathbf{P}$ be a pair of forces of equal magnitude but acting in the opposite directions. The two forces act at two points separated by a vector $\mathbf{r}$. The moment of the pair of forces is $\mathbf{M} = \mathbf{r} \times \mathbf{P}$. Recall how we calculate the cross product, and we write the components of the moment as

$$M_1 = r_2 P_3 - r_3 P_2,$$

$$M_2 = r_3 P_1 - r_1 P_3,$$

$$M_3 = r_1 P_2 - r_2 P_1.$$

Now we balance the moments of the forces acting on the six faces of the parallelepiped. Consider the pair of forces s_{i1} acting on two parallel faces. Because the two faces are sheared relative to each other, the two forces are not along the same line: the two forces have a moment. The two forces acting on points separated by the vector F_{i1}. The moment of the two forces is the cross product of the two vectors. The components of the moment are



$$F_{21} s_{31} - F_{31} s_{21},$$

$$F_{31} s_{11} - F_{11} s_{31},$$

$$F_{11} s_{21} - F_{21} s_{11}.$$

Forces on all six faces of the parallelepiped form three moments. The balance of moments requires that the sum of the moments vanish:

$$F_{2K} s_{3K} - F_{3K} s_{2K} = 0,$$

$$F_{3K} s_{1K} - F_{1K} s_{3K} = 0,$$

$$F_{1K} s_{2K} - F_{2K} s_{1K} = 0.$$

The summation is implied for the repeated index K, which represents the three pairs of faces. The above three equations can be written in a more compact form:

$$s_{iK} F_{jK} = s_{jK} F_{iK}.$$

This condition balances the moments acting on the parallelepiped. In general, neither the matrix s_{iK}, nor the matrix F_{jK}, is symmetric. However, the product $s_{iK} F_{jK}$ is a symmetric matrix.

THERMODYNAMICS

A pair of forces does work to a rod. When a rod elongates from length l to length $l+\delta l$, the force P does work $P\delta l$. The force and the length are work-conjugate.

Recall the definitions of stretch and the nominal stress:

$$l = \lambda L, \quad P = sA.$$

Consequently the work done by the force is

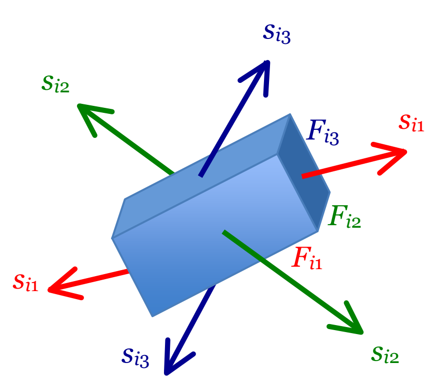
$$P\delta l = ALs\delta\lambda.$$

Since AL is the volume of the bar in the reference state, we note that

$$s\delta\lambda = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

The nominal stress and the stretch are work-conjugate.

A set of forces does work to a parallelepiped. We now generalize this definition to a body of an arbitrary shape undergoing a homogeneous deformation of an arbitrary type. Once again consider the homogenous deformation that changes a unit cube in the reference state to a parallelepiped in the current state. When the parallelepiped undergoes an infinitesimal, homogeneous deformation, and becomes a slightly different parallelepiped, one edge vector of the parallelepiped changes by δF_{i1}. Associated with this infinitesimal deformation, the pair of forces s_{i1} do work, which is calculated by the inner product of the two vectors, $s_{i1}\delta F_{i1}$. Similarly, another edge vector of the parallelepiped changes by δF_{i2}, and the pair of forces s_{i2} do work $s_{i2}\delta F_{i2}$. The third edge vector of the parallelepiped changes by δF_{i3}, and the pair of forces s_{i3} do work $s_{i3}\delta F_{i3}$. Associated with the infinitesimal, homogeneous deformation of the parallelepiped, the forces acting on the six faces of the parallelepiped do work:



$$s_{i1}\delta F_{i1} + s_{i2}\delta F_{i2} + s_{i3}\delta F_{i3}.$$

Using the summation convention, we write

$$s_{iK}\delta F_{iK} = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

The forces acting on the block can be applied by a set of hanging weights. Associated with the infinitesimal deformation, the potential energy of the weights changes by $-s_{iK}\delta F_{iK}$.

Free energy. The deformation is taken to be isothermal—that is, the body is in thermal equilibrium with a large reservoir of energy, held at a fixed temperature. Thus, we will not treat the temperature as a variable. You should have learned thermodynamics in a separate course. For a reminder of the isothermal process and the Helmholtz free energy, see http://imechanica.org/node/4878.

Let W be the Helmholtz free energy of the block in the current state, namely,

$$W = \frac{\text{Helmholtz free energy in the current state}}{\text{volume in the reference state}}.$$

The block and the weights together form a composite thermodynamic system. The Helmholtz free energy of the composite is the sum of that of the block and the potential energy of the weights:

$$W - s_{iK}F_{iK}.$$

The composite exchanges energy with the rest of the world by heat, but not by work.

Thermodynamic inequality. When the composite is in a state of equilibrium, the Helmholtz free energy of the composite is minimum. When the composite is not in a state of equilibrium, the Helmholtz free energy of the composite should only decrease. These statements are summarized as

$$\delta W - s_{iK}\delta F_{iK} \le 0.$$

The variation means the value of a quantity at a time minus that at a slightly earlier time. The inequality means that the increase in the free energy is no greater than the work done. As usual in thermodynamics, this inequality involves the direction of time, but not the duration of time. The thermodynamic inequality holds for arbitrary, infinitesimal, homogeneous deformation. The work done by the forces equals or exceeds the change in the free energy of the body. The difference is called the dissipation.

Thermodynamics does not prescribe a rheological model, but places a constraint in constructing a rheological model.

ELASTICITY

Thermodynamic equilibrium. As a particular rheological model, the block is pictured as a nonlinear spring. By this picture we mean that the block is assumed to be a reversible thermodynamic system. The block is in thermodynamic equilibrium as it deforms. The thermodynamic inequality is replaced with an equation:

$$\delta W - s_{iK}\delta F_{iK} = 0 .$$

In this model, the change in the Helmholtz free energy equals the work done by the forces. The thermodynamic equation holds for arbitrary infinitesimal deformation δF_{iK}.

The above statement implies that the Helmholtz free energy is a function of the deformation gradient:

$$W = W(\mathbf{F}).$$

According to the differential calculus, associated with an infinitesimal homogeneous deformation of the parallelepiped, the free energy changes by

$$\delta W = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}\delta F_{iK} .$$

Combining the two expressions, we obtain that

$$\left[\frac{\partial W(\mathbf{F})}{\partial F_{iK}} - s_{iK}\right]\delta F_{iK} = 0 .$$

Because this thermodynamic equation holds for arbitrary infinitesimal deformation δF_{iK}, we obtain that

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} .$$

Once the function $W(\mathbf{F})$ is prescribed, the above equation gives the stress-strain relation, or the equation of state.

Rigid-body rotation and balance of moments. The following two ideas are equivalent: the free energy of the block is invariant with respect to the rigid-body rotation and the moments acting on the block are balanced.

The free energy is invariant when the block undergoing a rigid-body rotation. Thus, the free energy depends on **F** through the deformation tensor **C**:

$$W = W(\mathbf{C}).$$

Recall that $C_{KL} = F_{iK}F_{iL}$ and $s_{iK} = \partial W / \partial F_{iK}$. We obtain that

$$s_{iK} = 2F_{iJ}\frac{\partial W(\mathbf{C})}{\partial C_{JK}} .$$

This equation readily confirms that

$$s_{iK}F_{jK} = s_{jK}F_{iK}.$$

This is the expression for the balance of the moments acting on the block.

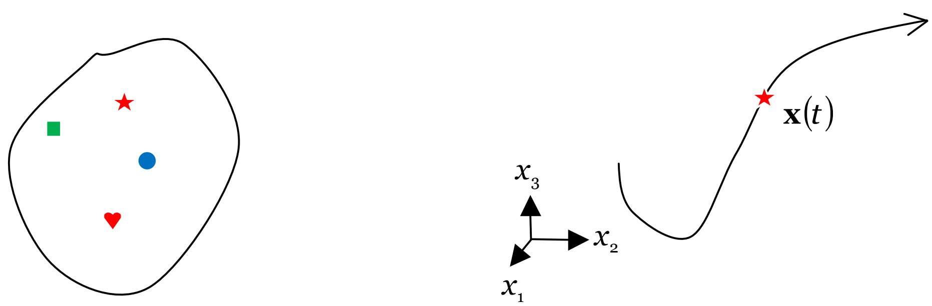
Models of elasticity. A model of elasticity represents a material by relating strain and stress when the material undergoes homogeneous deformation. To specify the model, we need to specify the nominal density of the Helmholtz free energy as a function of the Green deformation tensor:

$$W = W(\mathbf{C}).$$

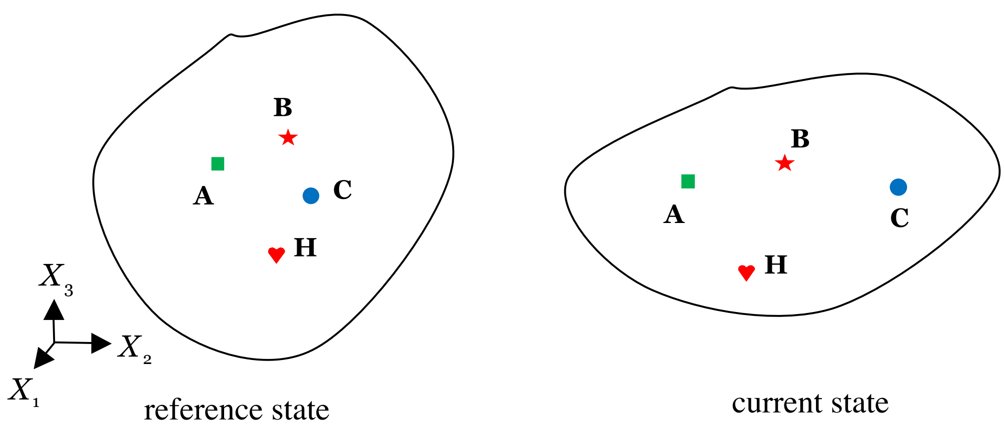
The tensor **C** is symmetric and has 6 independent components. Thus, to specify an elastic material model, we need to specify the free energy as a function of the 6 variables. For a given material, such a function is specified by a combination of experimental measurements and theoretical considerations. Trade off is made between the amount of effort and the need for accuracy. Commonly used free-energy functions are described in another set of notes (http://imechanica.org/node/14146).

INHOMOGENEOUS DEFORMATION

Name a material particle by the coordinate of the place occupied by the material particle when the body is in a reference state. Subject to loads, the body deforms in a three-dimensional space. Each small part of the space is called a place, labeled by its coordinate $\mathbf{x}$. Each small part of the body is called a material particle. At a given time t, the material particle occupies a place in the space. As time progresses, the material particle moves from one place to another. The trajectory of the material particle is described by the place of the particle as a function of time, $\mathbf{x}(t)$.



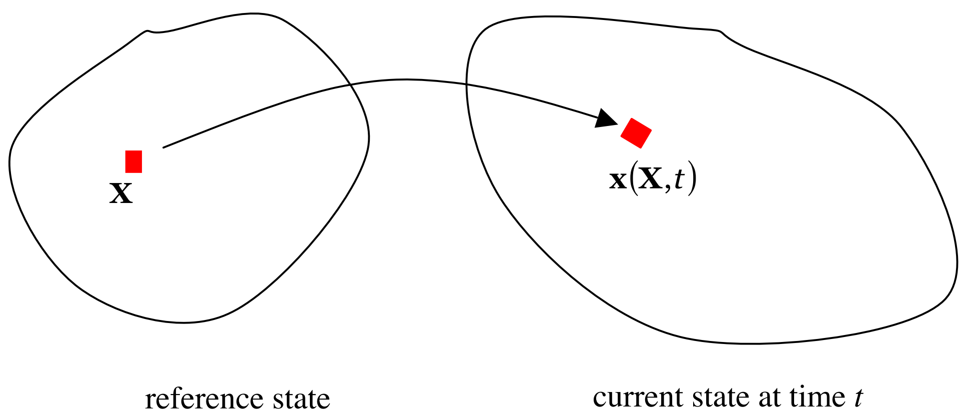
We can name a material particle any way we like. For example, we often name a material particle by using an English letter, a Chinese character, or a colored symbol—a red star for example. When dealing with a large number of material particles, we need a systematic scheme. For example, we name each material particle by the coordinate $\mathbf{X}$ of the place occupied by the material particle when the body is in a particular state.



We call this particular state of the body the *reference state*. We will use the phrase “the particle **X**” as shorthand for “the material particle that occupies the place with coordinate **X** when the body is in the reference state”. In addition to being systematic, naming material particles by coordinates has another merit. Once we know the name of one material particle, we know the names of all its neighbors, and we can apply calculus.

Often we choose the reference state to be the state when the body is unstressed. However, even without external loading, a body may be under a field of residual stress. Thus, we may not be able to always set the reference state as the unstressed state. Rather, any state of the body may be used as a reference state.

Indeed, the reference state need not be an actual state of the body, and can be a hypothetical state of the body. For example, we can use a flat plate as a reference state for a shell, even if the shell is always curved and is never flat. To enable us to use differential calculus, all that matters is that material particles can be mapped from the reference state to any actual state by a 1-to-1 smooth function.



As a body moves, every material particle in the body moves. Now we are given a body in a three-dimensional space. We have set up a system of coordinates in the space, and have chosen a reference state of the body to name material particles. When the body is in the reference state, a material particle occupies a place whose coordinate is **X**. At time t, the body deforms to a *current state*, and the material particle **X** moves to a place whose coordinate is **x**. The time-dependent field

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

describes the history of deformation of the body. The domain of this function is the coordinates of material particles when the body is in the reference state, as well as the time. The range of this function is the coordinates of the places occupied by the material particles. The function $\mathbf{x}(\mathbf{X}, t)$ gives the trajectory of every material particle in the body. A central aim of continuum mechanics is to evolve the field of deformation $\mathbf{x}(\mathbf{X}, t)$ by developing an equation of motion.

The function $\mathbf{x}(\mathbf{X}, t)$ has two independent variables: $\mathbf{X}$ and t. The two variables can change independently. We next examine their changes separately.

Exercise. Give a pictorial interpretation of the following field of deformation:

$$x_1 = X_1 + X_2 \tan \gamma(t)$$
$$x_2 = X_2$$
$$x_3 = X_3$$

Compare the above field with another field of deformation:

$$x_1 = X_1 + X_2 \sin \gamma(t)$$
$$x_2 = X_2 \cos \gamma(t)$$
$$x_3 = X_3$$

Displacement, velocity, and acceleration of a material particle. At time t, the material particle $\mathbf{X}$ occupies the place $\mathbf{x}(\mathbf{X}, t)$. At a slightly later time $t + \delta t$, the same material particle $\mathbf{X}$ occupies a different place $\mathbf{x}(\mathbf{X}, t + \delta t)$. During the short time between t and $t + \delta t$, the material particle $\mathbf{X}$ moves by a small displacement:

$$\delta \mathbf{x} = \mathbf{x}(\mathbf{X}, t + \delta t) - \mathbf{x}(\mathbf{X}, t).$$

The velocity of the material particle $\mathbf{X}$ at time t is defined as

$$\mathbf{v} = \frac{\mathbf{x}(\mathbf{X}, t + \delta t) - \mathbf{x}(\mathbf{X}, t)}{\delta t},$$

or,

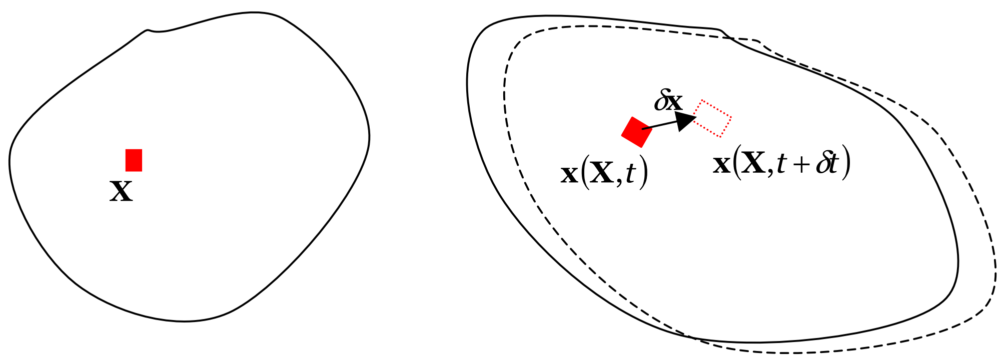
$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}.$$

The velocity is a time-dependent field, $\mathbf{v}(\mathbf{X}, t)$.

The acceleration of the material particle $\mathbf{X}$ at time t is

$$\mathbf{a}(\mathbf{X}, t) = \frac{\partial^2 \mathbf{x}(\mathbf{X}, t)}{\partial t^2}.$$

The fields of velocity and acceleration are linear in the field of deformation $\mathbf{x}(\mathbf{X},t)$.



Exercise. Given a field of deformation,

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

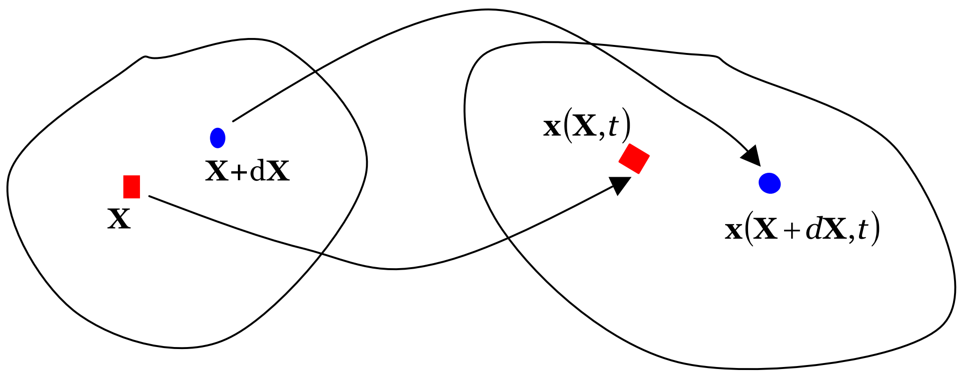
Calculate the fields of velocity and acceleration.

Deformation gradient. We have just interpreted the partial derivative of the function $\mathbf{x}(\mathbf{X},t)$ with respect to *t*. We next interpret the partial derivative of the function $\mathbf{x}(\mathbf{X},t)$ with respect to $\mathbf{X}$.

Consider two nearby material particles in the body. When the body is in the reference state, the first particle occupies the place with the coordinate $\mathbf{X}$, and the second particle occupies the place with the coordinate $\mathbf{X}+d\mathbf{X}$. The vector $d\mathbf{X}$ connects the places occupied by the two material particles when the body is in the reference state. When the body is in the current state at time *t*, the first material particle occupies the place with the coordinate $\mathbf{x}(\mathbf{X},t)$, and the second material particle occupies the place with the coordinate $\mathbf{x}(\mathbf{X}+d\mathbf{X},t)$. At time *t*, the two material particles are ends of a vector:

$$d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t).$$

Note the difference between two ideas: $\delta\mathbf{x} = \mathbf{x}(\mathbf{X},t+\delta t) - \mathbf{x}(\mathbf{X},t)$ means the displacement of any single material particle at two different times, and $d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t)$ means the distance between two material particles at a given time.



The Taylor expansion of the function $\mathbf{x}(\mathbf{X},t)$ is

$$x_i(\mathbf{X}+d\mathbf{X},t) = x_i(\mathbf{X},t) + \frac{\partial x_i(\mathbf{X},t)}{\partial X_K} dX_K .$$

Here the time t is fixed, and only the term linear in dX_K is retained. The expansion is accurate when the two material particles are sufficiently close to each other—that is, when the vector $d\mathbf{X}$ is sufficiently short.

Rewrite the Taylor expansion as

$$dx_i = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K} dX_K .$$

Recall the definition of the deformation gradient for a homogeneous deformation. For a time dependent, homogeneous deformation, the deformation gradient $\mathbf{F}$ is defined as the linear map from $d\mathbf{X}$ in the reference state to $d\mathbf{x}$ in the current state, namely,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} .$$

A comparison of the two expressions identifies that

$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K}.$$

The field $\mathbf{F}(\mathbf{X},t)$ is the gradient of the field of deformation $\mathbf{x}(\mathbf{X},t)$. Thus, when a body undergoes a time-dependent, inhomogeneous deformation, at a fixed time a set of material particles near one another behaves just like homogeneous deformation. Within this set of material particles, a straight segment of material particles in the reference state remains a straight segment in the current state, but is stretched and rotated. The deformation gradient $\mathbf{F}$ maps the segment from the reference state to the current state, and is given by the gradient of the field $\mathbf{x}(\mathbf{X},t)$.

Exercise. Given a field of deformation,

$$x_1 = X_1 + X_2 \sin\gamma(t)$$
$$x_2 = X_2 \cos\gamma(t)$$
$$x_3 = X_3$$

Calculate the deformation gradient. Is the deformation homogeneous?

CONSERVATION OF MASS

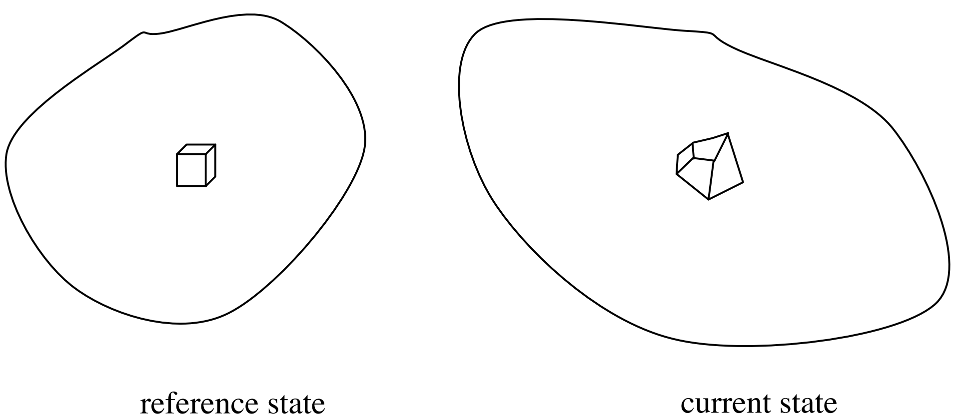
When a body is in the reference state, a material particle occupies a place with coordinate **X**. Consider a material element of volume around the particle. When the body is in a current state at t, the same material element deforms to some other shape. Let ρ_R be the nominal density of mass, namely,

$$\rho_R = \frac{\text{mass of the material element in current state}}{\text{volume of the material element in reference state}}.$$

During deformation, we assume that the material element does not gain or lose mass, so that the nominal density of mass, ρ_R, is time-independent. If the body in the reference state is inhomogeneous, the nominal density of mass in general varies from one material particle to another. Combining these two considerations, we write the nominal density of mass as a function of material particle:

$$\rho_R = \rho_R(\mathbf{X}).$$

This function is given as an input to our theory. The conservation of mass requires that the nominal density of mass is independent of time.



CONSERVATION OF MOMENTUM

Body force. Consider a material element of volume around material particle $\mathbf{X}$. When the body in the reference state, the volume of the element is $dV(\mathbf{X})$. When the body is in the current state at time t, the force acting on the element is denoted by $\mathbf{B}(\mathbf{X},t)dV(\mathbf{X})$, namely,

$$\mathbf{B}(\mathbf{X},t) = \frac{\text{force in current state}}{\text{volume in reference state}}.$$

The force $\mathbf{B}(\mathbf{X},t)dV(\mathbf{X})$ is called the *body force*, and the vector $\mathbf{B}(\mathbf{X},t)$ the nominal density of the body force. The body force is applied by an agent external to the body.

Exercise. Given a field of deformation:

$$x_1 = X_1 + X_2 \sin \gamma(t)$$
$$x_2 = X_2 \cos \gamma(t)$$
$$x_3 = X_3$$

The body is in a gravitational field, and is of nominal density of 1000kg/m^3. The gravitational field is pointing down along the x_2 axis. Calculate the nominal density of the body force due to gravitation in the current state at time t.

Inertial force. The dynamics of a particle is governed by Newton's second law:

Force = Mass times Acceleration.

From this expression, we can regard the term "mass times acceleration" as the inertial force, acting in the direction opposite to that of the acceleration. Newton's second law is then expressed as the balance forces acting on the particle, including the inertial force.

For a continuum body, we can regard the term

$$-\rho_R(\mathbf{X})\frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}$$

as a special type of the body force, called the inertial force.

Balance of forces. The dynamics of a particle is governed by Newton's second law:

Force = Mass times Acceleration.

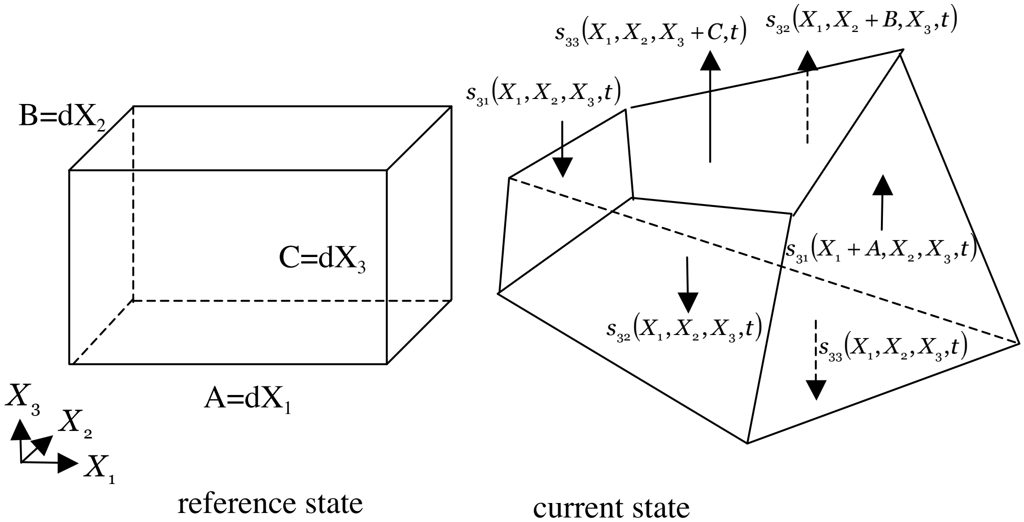
We now apply this law to a small block in a body. In the reference state, the block is rectangular, with faces parallel to the coordinate planes, and of sides A, B, C.

In the current state, the block deforms to some other shape. In the free-body diagram of the block in the current state, we should include surface forces, body forces, and inertial forces. (For clarity, in the figure only surface forces in direction 3 on the six faces of the block are indicated.) Newton's second law states that

$$
\begin{aligned}
& BCs_{i1}\left(X_1 + A, X_2, X_3, t\right) - BCs_{i1}\left(X_1, X_2, X_3, t\right) \\
& + CAs_{i2}\left(X_1, X_2 + B, X_3, t\right) - CAs_{i2}\left(X_1, X_2, X_3, t\right) \\
& + ABs_{i3}\left(X_1, X_2, X_3 + C, t\right) - ABs_{i3}\left(X_1, X_2, X_3, t\right) \\
& + ABCB_i\left(\mathbf{X}, t\right) \\
& = ABC\rho_R\left(\mathbf{X}\right)\frac{\partial^2 x_i\left(\mathbf{X}, t\right)}{\partial t^2}
\end{aligned}
$$

Dividing this equation by the volume of the block, *ABC*, we obtain that

$$\frac{\partial s_{iK}\left(\mathbf{X}, t\right)}{\partial X_K} + B_i\left(\mathbf{X}, t\right) = \rho_R\left(\mathbf{X}\right)\frac{\partial^2 x_i\left(\mathbf{X}, t\right)}{\partial t^2}.$$



Divergence theorem. In the following development, we will need the divergence theorem:

$$\int \frac{\partial f\left(\mathbf{X}\right)}{\partial X_K} dV = \int f N_K \, dA \,,$$

where $f\left(\mathbf{X}\right)$ is a field, N_K is unit vector normal to the surface. The integrals are over the volume of a region and the surface of a region, respectively.

An alternative approach to the balance of linear momentum. Consider an arbitrary part of the body. In the current state, the combined forces acting on the part must vanish:

$$\int s_{iK} N_K \, dA + \int \left[B_i - \rho_R(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} \right] dV = 0 .$$

The first integral is over the surface of the part in the reference state, and the second integral is over the volume of the part in the reference state.

According to the divergence theorem, we write

$$\int s_{iK} N_K \, dA = \int \frac{\partial s_{iK}}{\partial X_K} dV .$$

A combination of the above expressions gives that

$$\int \left[\frac{\partial s_{iK}}{\partial X_K} + B_i - \rho_R(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} \right] dV = 0 .$$

This equality holds for arbitrary part of the body. Consequently, the integrand must vanish.

Conservation of angular momentum. Consider an arbitrary part of the body. In the current state, the combined moments acting on the part must vanish:

$$\int x_j s_{iK} N_K \, dA + \int x_j \left[B_i - \rho_R(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2} \right] dV$$

$$= \int x_i s_{jK} N_K \, dA + \int x_i \left[B_j - \rho_R(\mathbf{X}) \frac{\partial^2 x_j(\mathbf{X},t)}{\partial t^2} \right] dV$$

According to the divergence theorem, we write

$$\int x_j s_{iK} N_K \, dA = \int \frac{\partial (x_j s_{iK})}{\partial X_K} dV = \int \left(F_{jK} s_{iK} + x_j \frac{\partial s_{iK}}{\partial X_K} \right) dV .$$

We can similarly convert $\int x_i s_{jK} N_K \, dA$ to an integral over volume. Combining the above expressions and using the conservation of linear momentum, we obtain that

$$\int F_{jK} s_{iK} \, dV = \int F_{iK} s_{jK} \, dV .$$

This equality holds for arbitrary part of the body. Consequently, the integrands must be equal:

$$F_{jK} s_{iK} = F_{iK} s_{jK} .$$

This expression recovers what we know before. Thus, the conservation of angular momentum leads to no new equations.

Weak statement of the balance of forces. The balance of forces results in two equations:

$$T_i = s_{iK} N_K ,$$

$$\frac{\partial s_{iK}}{\partial X_K} + B_i = \rho_R \frac{\partial^2 x_i}{\partial t^2} .$$

This pair of equations may be called the *strong statement* of the balance of forces.

Denote an arbitrary field by $\Delta_i(\mathbf{X})$. Multiplying $\Delta_i(\mathbf{X})$ to the two equations of the strong statement, integrating over the surface of the part and the volume of the part, respectively, and then adding the two, we obtain that

$$\int \left(\frac{\partial s_{iK}}{\partial X_K} + B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i \, dV + \int \left(T_i - s_{iK} N_K \right) \Delta_i \, dA = 0 .$$

The integrals extend over the volume and the surface of the body.

Manipulate one term in the above equation:

$$\int \frac{\partial s_{iK}}{\partial X_K} \Delta_i \, dV = \int \frac{\partial \left(s_{iK} \Delta_i \right)}{\partial X_K} dV - \int s_{iK} \frac{\partial \Delta_i}{\partial X_K} dV$$

$$= \int s_{iK} \Delta_i N_K \, dA - \int s_{iK} \frac{\partial \Delta_i}{\partial X_K} dV$$

In the above, we have used the divergence theorem.

Consequently, we obtain that

$$\int s_{iK} \frac{\partial \Delta_i}{\partial X_K} dV = \int T_i \Delta_i \, dA + \int \left(B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i \, dV .$$

The strong statement of the balance of forces implies that the above equation holds for any field $\Delta_i(\mathbf{X})$. This statement is known as the *weak statement* of the balance of forces. The field $\Delta_i(\mathbf{X})$ is called a *test function*.

Exercise. Start with the weak statement of the balance of forces, and show that the weak statement implies the strong statement.

THERMODYNAMICS OF INHOMOGENEOUS DEFORMATION

For homogeneous deformation, thermodynamics dictates that the increase in the free energy is no greater than the work done

$$\delta W \le s_{iK}\delta F_{iK} .$$

The variation means the value of a quantity at a time minus that at a slightly earlier time. The inequality involves the direction of time, but not the duration of time. The thermodynamic inequality holds for arbitrary infinitesimal, homogeneous deformation.

We now confirm that so long as the material model satisfies the thermodynamic inequality, the inhomogeneous, time-dependent deformation also satisfies thermodynamics. Interpret the test function as a virtual change in the deformation, $\Delta_i(\mathbf{X}) \to \delta x_i(\mathbf{X})$, so that $\partial \Delta_i(\mathbf{X})/\partial X_K \to \delta F_{iK}$ and the weak statement becomes

$$\int s_{iK}\,\delta F_{iK} dV = \int T_i\,\delta x_i\,dA + \int\left(B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2}\right)\delta x_i\,dV .$$

This statement is known as the principle of virtual work. The virtual deformation $\delta x_i(\mathbf{X})$ is unrelated to the actual deformation $x_i(\mathbf{X},t)$.

If the material model satisfies the thermodynamic inequality, $\delta W \le s_{iK}\delta F_{iK}$, a combination of the inequality and the principle of virtual work gives that

$$\int \delta W dV \le \int T_i\,\delta x_i\,dA + \int\left(B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2}\right)\delta x_i\,dV .$$

This is the thermodynamic inequality for time-dependent, inhomogeneous deformation. The left-hand side is the increment of the free energy of the body, and the right-hand side is the work done by the applied forces and the inertial force. Thus, once the material model satisfies the thermodynamic inequality, $\delta W \le s_{iK}\delta F_{iK}$, the entire body also satisfies the thermodynamic model.

INITIAL AND BOUNDARY VALUE PROBLEMS IN ELASTICITY

The ingredients of finite elasticity. *Model of elasticity*. We describe the state of each material particle by two tensors. The deformation gradient $\mathbf{F}$ is the linear operator that maps a segment of a line of material particles in the reference state to a straight segment in the current state, $\mathbf{y} = \mathbf{FY}$. The nominal stress $\mathbf{s}$ is the linear operator that maps an area vector of a planar region of material particles in the reference state to the force acting on the same region of material particles in the current state, $\mathbf{P} = \mathbf{sA}$. We specify the model of elasticity by giving the Helmholtz free-energy function $W(\mathbf{F})$. The nominal stress relates to the deformation gradient by

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

The function $W(\mathbf{F})$ depends on $\mathbf{F}$ through the product $\mathbf{F}^T\mathbf{F}$. Such a material model conserves angular momentum, $F_{jK}s_{iK} = F_{iK}s_{jK}$.

Compatibility of deformation. A body is represented by a set of material particles. Each material particle is named by its place $\mathbf{X}$ when the body is in a reference state. In the current state at time *t,* the material particle occupies the place with coordinate $\mathbf{x}$. The function $\mathbf{x}(\mathbf{X},t)$ describes the deformation of the entire body in time. The deformation gradient relates to the deformation function through

$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K}.$$

Balance of forces. The body is prescribed with a field of mass density, $\rho_R(\mathbf{X})$. The body is subject to a field of body force $\mathbf{B}(\mathbf{X},t)$. The balance of forces requires that

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = \rho_R(\mathbf{X})\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2}.$$

On part of the surface of the body, the traction is prescribed, so that

$$s_{iK}(\mathbf{X},t)N_K(\mathbf{X}) = \text{prescribed}.$$

Exercise. We will be restricted to isothermal processes. The body is in thermal contact with a reservoir of energy held at a fixed temperature. This temperature is also assumed to be held in the body. In the isothermal process, the principle of the conservation of energy is not used in formulating the continuum theory. Why?

Initial and boundary value problems. We can combine the three equations and write

$$\frac{\partial}{\partial X_K}\left(\frac{\partial W(\mathbf{F})}{\partial F_{iK}}\right) + B_i(\mathbf{X},t) = \rho_R(\mathbf{X})\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2}.$$

This equation, known as the equation of motion, is the partial differential equation that evolves the field of deformation $\mathbf{x}(\mathbf{X},t)$ in time, subject to the following initial and boundary conditions.

Initial conditions are given by prescribing at time t_o the places of all the particles, $\mathbf{x}(\mathbf{X},t_o)$, and the velocities of all the particles, $\mathbf{v}(\mathbf{X},t_o)$.

For every material particle on the surface of the body, we prescribe either one of the following two boundary conditions. On part of the surface of the body, S_t, the traction is prescribed, so that

$$s_{iK}(\mathbf{X},t)N_K(\mathbf{X}) = \text{prescribed}, \quad \text{for } \mathbf{X}\in S_t.$$

On the other part of the surface of the body, S_u, the position is prescribed, so that

$$\mathbf{x}(\mathbf{X},t) = \text{prescribed}, \quad \text{for } \mathbf{X}\in S_u.$$

Now we have the basic equations. What do we do next? The above formulation of the boundary-value problem has existed for well over a century. However, exploration of its consequences remains active to this day. Representative activities include

- Model a specific elastic material by constructing a function $W(\mathbf{F})$, by a combination of microcosmic modeling and experimental testing.
- Model a specific phenomenon of elastic deformation by formulating a boundary-value problem.
- Analyze such a boundary-value problem by analytic techniques, such as dimensional analysis and linear perturbation.
- Analyze such a boundary-value problem by numerical methods, such as using commercial finite element package.

Of course, you can also play another kind of game: you can use the similar approach to formulate models for phenomena other than the deformation of an elastic body.

Finite element method. The weak statement of the balance of forces is the basis for the finite element method (http://imechanica.org/node/324). Here I sketch the basic ideas. These ideas are greatly amplified in a separate course on finite element method.

In the weak statement, the volume integrals extend over the entire body, and the surface integral extends over the part of the surface of the body on which traction is prescribed. The test function $\Delta_i(\mathbf{X})$ is set to vanish on part of the surface where displacement is prescribed. In addition to the weak statement of the balance of forces, three other ingredients of the boundary value problem are the conservation of mass:

$$\rho_R = \rho_R(\mathbf{X}),$$

the compatibility of deformation:

$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K},$$

and the model of material

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

The functions $\rho_R(\mathbf{X})$ and $W(\mathbf{F})$ are prescribed.

Let $N^\alpha(\mathbf{X})$ be a set of functions defined in the body, known as the shape functions. Interpolate the test function in terms of the shape functions:

$$\Delta_i(\mathbf{X}) = \sum_\alpha D_i^\alpha N^\alpha(\mathbf{X}).$$

This interpolation represents the test functions in terms of a set of discrete values D_i^α. Because the test function is arbitrary, the values D_i^α are also arbitrary.

Insert this expression into the weak statement, and we obtain that

$$\int s_{iK} \sum_\alpha \left(D_i^\alpha \frac{\partial N^\alpha}{\partial X_K} \right) dV = \int T_i \sum_\alpha \left(D_i^\alpha N^\alpha \right) dA + \int \left(B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2} \right) \sum_\alpha \left(D_i^\alpha N^\alpha \right) dV.$$

The weak statement holds for arbitrary values of D_i^α. This single equation is equivalent to a set of equations:

$$\int s_{iK} \frac{\partial N^\alpha}{\partial X_K} dV = \int T_i N^\alpha \, dA + \int \left(B_i - \rho_R \frac{\partial^2 x_i}{\partial t^2} \right) N^\alpha \, dV.$$

The displacement is also interpolated using the same shape functions:

$$x_i(\mathbf{X},t) - X_i = \sum_\alpha u_i^\alpha(t) N^\alpha(\mathbf{X}).$$

The compatibility of deformation and the material model are used to express the stress in terms of $u_i^\alpha(t)$. Consequently, the weak statement becomes a set of ordinary differential equations for $u_i^\alpha(t)$. This set of ordinary differential equations is evolved using computer.

State of equilibrium. Subject to a static load, a body can reach a state of equilibrium. In the state of equilibrium, the deformation of the body no longer changes with time, and the field of deformation is time-independent, described by the function $\mathbf{x}(\mathbf{X})$. This function characterizes the state of equilibrium.

The field of deformation function $\mathbf{x}(\mathbf{X})$ is governed by the field equations

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K},$$

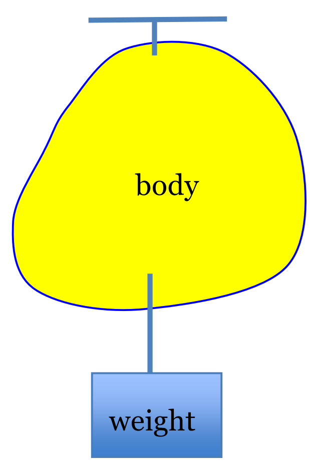
$$\frac{\partial s_{iK}(\mathbf{X})}{\partial X_K} + B_i(\mathbf{X}) = 0,$$

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}},$$

as well as time-independent boundary conditions. These equations result in a boundary-value problem.

Exercise. Develop the basic equations for the finite element method to solve static elasticity problems.

Stability of a state of equilibrium. A body is subject to a static load. The static load may be represented by an idealized loading device, such as a dead weight or a constant pressure. The body and the loading device together form a composite thermodynamic system. This composite system interacts with the rest of the world by heat transfer, but not by work. A state of equilibrium is stable if the state minimizes the Helmholtz free energy of the composite; see separate notes for a reminder of free energy (http://imechanica.org/node/4878).



The Helmholtz free energy of the composite is the sum of the Helmholtz free energy of the body and that of the loading device. For example, if the static load is a dead weight, the Helmholtz free of the loading device is simply the potential energy of the dead weight, -*Pl*, where *P* is the weight (i.e., the fixed force) and *l* is the displace of the weight. The free energy of the composite is

$$\int W(\mathbf{F})dV - Pl,$$

where the deformation gradient relates to the field of deformation by

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K}.$$

The Helmholtz free energy is a functional of the field of deformation, $\mathbf{x}(\mathbf{X})$. Of all possible field of deformation, a stable state of equilibrium minimizes the Helmholtz free energy of the composite.

Exercise. A body is subject to a constant pressure on its surface. The body is made of an elastic material characterized by the nominal density of Helmholtz free energy as a function of the deformation gradient, $W(\mathbf{F})$. Picture that the pressure is applied by some device. The body and the device together form a composite system. Write the Helmholtz free energy of the composite.

ALTERNATIVE MATHEMATICAL REPRESENTATIONS

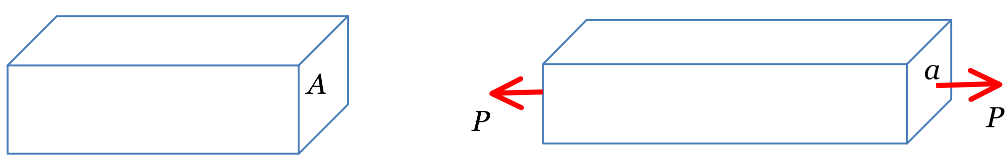
An idea in the continuum theory often has alternative mathematical representations. The alternative representations add no substance to the theory, but they appear in the literature so frequently that you should know them. Besides, alternative representations of an idea may shed light on the idea itself. Here we give a few examples. You can find many more in textbooks.

True stress. Subject to an axial force, the rod changes its length and cross-sectional area. The true stress is defined by

$$\text{true stress} = \frac{\text{force in current state}}{\text{area in current state}}.$$

In the current state, let P be the axial force, and a be the cross-sectional area of the rod. The true stress is defined by

$$\sigma = \frac{P}{a}.$$



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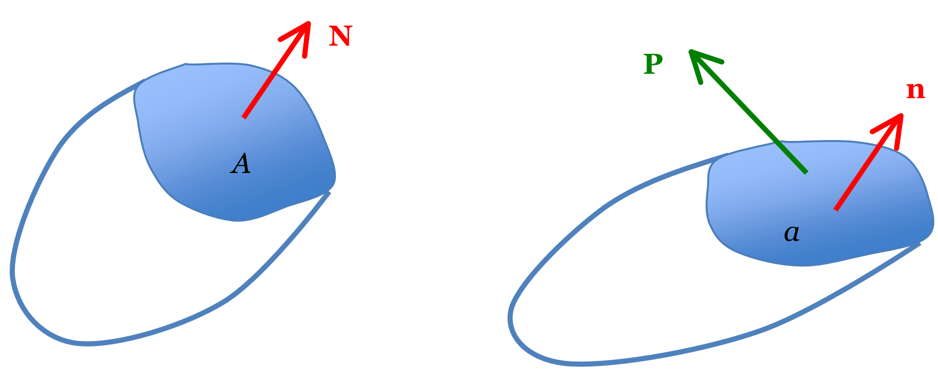

reference state current state

We now generalize this definition to a body of an arbitrary shape undergoing a homogeneous deformation of an arbitrary type. Consider a set of material particles. When the body is in the current state, the set of material particles lies on a plane of unit normal vector **n**, in a region of area a. The vector $a\mathbf{n}$ represents the planar region as a vector, written as $\mathbf{a} = a\mathbf{n}$. Acting on the planar region is a force **P**.

Define the true stress as the operator operator σ that maps the area vector **a** to the force **P**:

$$\mathbf{P} = \sigma(\mathbf{a}).$$

The true stress is also known as the Cauchy stress. The true stress is also a linear operator.

The components of true stress. The components of the true stress form a matrix. Consider a unit cube in the body in the current state. Each column of the matrix is the force acting on a face of the cube.

Stress-traction relation. In the current state, consider a plane of unit normal vector **n**. Define the true traction **t** as force acting on the plane divided by the area of the plane. Consider a tetrahedron formed by the plane and the three coordinate planes. The balance of forces acting on the tetrahedron requires that

$$t_i = \sigma_{ij} n_j.$$

The true stress maps one vector (the unit normal vector) to another vector (the true traction).

Balance of moments. For a body in the current state, imagine a unit cube in the body orientated in the directions of a set of rectangular coordinates. The true stress σ_{ij} is defined as the force in direction i acting on a face of the cube of normal vector in direction j. True stress is also known as the Cauchy stress. The balance of momentum requires that

$$\sigma_{ij} = \sigma_{ji}.$$

Relation between true stress and nominal stress. Consider a set of material particles. In the reference state, the set of material particles forms a region represented by the area vector **A**. After a homogeneous deformation **F**, the same set of material particles forms a region represented by the vector **a**. The two area vectors are related by the formula of Nanson, $\mathbf{F}^T\mathbf{a} = J\mathbf{A}$.

We have expressed the force acting on the plane in the current state, **P**, in two ways: $\mathbf{P} = \sigma(\mathbf{a})$ and $\mathbf{P} = \mathbf{s}(\mathbf{A})$. Thus, $\sigma(\mathbf{a}) = \mathbf{s}(\mathbf{A})$. This expression, along with the formula of Nanson, becomes

$$\sigma(\mathbf{a}) = \frac{\mathbf{s}(\mathbf{F}^T\mathbf{a})}{J}.$$

The expression equates two vectors. Because the equation holds for arbitrary choice of **a**, we reach an expression that equates two tensors:

$$\sigma = \frac{\mathbf{sF}^T}{J}.$$

Write this expression in terms of components:

$$\sigma_{ij} = \frac{s_{iK}F_{jK}}{J}.$$

This expression relates the true stress and the nominal stress.

We have expressed the balance of moments in terms of the true stress $\sigma_{ij} = \sigma_{ji}$, as well as in terms of the nominal stress, $s_{iK}F_{jK} = s_{jK}F_{iK}$. These two expressions are equivalent.

The second Piola-Kirchhoff stress. Here is another redundant idea commonly in use. Define the second Piola-Kirchhoff stress, S_{KL}, by

$$\frac{\text{work in the curent state}}{\text{volume in the reference state}} = S_{KL}\delta E_{KL}.$$

This expression defines a new measure of stress, S_{KL}. Because E_{KL} is a symmetric tensor, we can set S_{KL} to be symmetric.

Recall that we have also expressed the same work by $s_{iK}\delta F_{iK}$. Equating the two expressions for work, we write

$$s_{iK}\delta F_{iK} = S_{KL}\delta E_{KL}.$$

Recall the definition of the Lagrange strain,

$$E_{KL} = \frac{1}{2}\left(F_{iK}F_{iL} - \delta_{KL}\right).$$

We obtain that

$$\delta E_{KL} = \frac{1}{2}\left(F_{iL}\delta F_{iK} + F_{iK}\delta F_{iL}\right)$$

and

$$s_{iK}\delta F_{iK} = S_{KL}F_{iL}\delta F_{iK}.$$

Here we have used the symmetry $S_{KL} = S_{LK}$. In the above equation, each side is a sum of nine terms. Each component of δF_{iK} is an arbitrary and independent variation. Consequently, the factors in front of each component of δF_{iK} must equal, giving

$$s_{iK} = S_{KL}F_{iL}.$$

This equation relates the first Piola-Kirchhoff stress to the second Piola-Kirchhoff stress.

Exercise. For an elastic material, the nominal density of the Helmholtz free energy is a function of the Lagrange strain, $W(\mathbf{E})$. Starting from basic definitions and thermodynamic considerations, show that the second Piola-Kirchhoff stress relates to the Lagrange strain as

$$S_{KL} = \frac{\partial W(\mathbf{E})}{\partial E_{KL}}.$$

EVOLVING HOMOGENEOUS DEFORMATION IN TIME

A rod elongates as a function of time. A rod is of length L in a reference state, and is of length $l(t)$ at time t. The length of the rod in the reference state is of course independent of time. The stretch is defined by

$$\lambda = \frac{l(t)}{L}.$$

When the length of a bar changes by a small amount from l to $l + dl$, the increment in the natural strain is defined as

$$d\varepsilon = \frac{dl}{l}.$$

Thus, the rate of the natural strain is

$$\frac{d\varepsilon}{dt} = \frac{dl(t)}{ldt}.$$

Evolve homogeneous deformation in time. As time goes on, a body undergoes a succession of homogeneous deformations, represented by the deformation gradient as a function of time, $\mathbf{F}(t)$. Consider a set of material particles in the body. When the body is in the reference state, the set of material particles form a straight segment, which we denote by a vector $\mathbf{Y}$. When the body is in the current state at time t, the same set of material particles forms another straight segment, which we denote by vector $\mathbf{y}(t)$. The deformation from the reference state to the current state is homogeneous, so that the two vectors are related to each other through a linear map:

$$\mathbf{y}(t) = \mathbf{F}(t)\mathbf{Y}.$$

The deformation gradient $\mathbf{F}(t)$ changes with time. At a given time, the same deformation gradient maps any straight segment in the reference state to the straight segment in the current state.

Note that the vector $\mathbf{Y}$ is independent of time, so that

$$\frac{d\mathbf{y}(t)}{dt} = \frac{d\mathbf{F}(t)}{dt}\mathbf{Y}.$$

The quantity $d\mathbf{y}(t)/dt$ is the rate of change of the straight segment of the same set of material particles.

Velocity gradient. A body undergoes a time-dependent, homogeneous deformation. A material particle in the body is at position $\mathbf{X}$ in the reference state, and is at position $\mathbf{x}$ in the current state at time t. The velocity of the material particle is

$$\mathbf{v} = \frac{d\mathbf{x}(t)}{dt}.$$

If the body undergoes a rigid-body translation, all material particles in the body move by the same velocity. If the body also rotates and stretches, however, different material particles in the body can move by different velocities. Consider another material particle, whose position is $\mathbf{X}_o$ in the reference state and is $\mathbf{x}_o$ in the current state. The displacement of this material particle is

$$\mathbf{v}_o = \frac{d\mathbf{x}_o(t)}{dt}.$$

Define a new tensor **L** by

$$\mathbf{v} - \mathbf{v}_o = \mathbf{L}(\mathbf{x} - \mathbf{x}_o)$$

The tensor **L** is called the *velocity gradient*. In general, the tensor is time-dependent, $\mathbf{L}(t)$.

The distance between the two material particles in the current state is $\mathbf{y} = \mathbf{x} - \mathbf{x}_o$. The definition of the velocity gradient is equivalent to

$$\frac{d\mathbf{y}}{dt} = \mathbf{L}\mathbf{y}.$$

Exercise. Show that

$$\mathbf{LF} = \frac{d\mathbf{F}(t)}{dt}.$$

Given the deformation as a function of time, $\mathbf{F}(t)$, the above expression calculates the rate of deformation $\mathbf{L}(t)$.

Rate of deformation. Consider a set of material particles. At time t, the set of material particles forms a rectangular block. At time $t + dt$, the same set of material particles forms a parallelepiped. The rates of normal strain are

$$L_{11}, L_{22}, L_{33}.$$

The rates of shear strain are $L_{12} + L_{21}, L_{23} + L_{32}, L_{31} + L_{13}$. These six quantities do not form a tensor. However, we can define a tensor as

$$D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right).$$

This definition is also written as

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right).$$

The tensor **D** is symmetric and is known as the rate-of-deformation tensor. The rate of deformation **D** is the symmetric part of the velocity gradient **L**.

Stretching a line of material particles. Consider a set of material particles. At time *t*, the set of material particles forms a segment of a straight line. The homogeneous deformation of the body translates, rotates and stretches the segment, but the same set of material particles remains a straight segment at all time. Represent the segment at time *t* by a vector $\mathbf{y}(t)$. The vector obeys

$$\frac{d\mathbf{y}}{dt} = \mathbf{L}\mathbf{y} .$$

Take dot product of the above equation with **y**, and we obtain that

$$\frac{d(\mathbf{y}\cdot\mathbf{y})}{dt} = 2\mathbf{y}^T\mathbf{D}\mathbf{y} .$$

The above expression only contains the symmetric part of the velocity gradient. Write the straight segment as $\mathbf{y} = l\mathbf{m}$, where *l* is the length of the segment, and **m** is the unit vector along the segment. The above equation is written as

$$\frac{dl}{ldt} = 2\mathbf{m}^T\mathbf{D}\mathbf{m} .$$

This expression calculates the rate of natural strain of the line of material particles.

Spin. Denote the anti-symmetric part of the rate of deformation by

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right).$$

This tensor is known as the spin. The spin does not distort the material, and does not cause any stress.

Power. Consider a unit cube in the current state. The velocity gradients L_{ij} represent the velocity of one face of the cube relative to another, and the true stresses σ_{ij} represent forces acting on the faces. We obtain that

$$\frac{\text{power in the curent state}}{\text{volume in the current state}} = \sigma_{ij}L_{ij} .$$

Because the true stress is a symmetric tensor, the above expression is equivalent to

$$\frac{\text{power in the curent state}}{\text{volume in the current state}} = \sigma_{ij} D_{ij} .$$

Exercise. Recall the expression

$$\frac{\text{work in the curent state}}{\text{volume in the reference state}} = s_{iK} \delta F_{iK} .$$

Note that the nominal stress is related to the true stress, and the deformation gradient is related to the displacement gradient. Confirm that the above expression is consistent with the expression for the power density.

Viscosity. Viscous flow is a material model with the following assumptions. The Helmholtz free energy does not change with deformation, and the stress is a function of the rate of deformation. Thermodynamics requires that

$$\sigma_{ij} D_{ij} > 0$$

for any non-zero stress state.

A special case is the Newtonian fluids. The components of the stress relate to the components of the rate of deformation as

$$\sigma_{ij} = 2\eta D_{ij} - p\delta_{ij} ,$$

where η is the viscosity. The material is assumed to be compressible:

$$D_{kk} = 0 .$$

EULERIAN FORMULATION OF INHOMOGENEOUS DEFORMATION

At time t, a material particle $\mathbf{X}$ moves to position $\mathbf{x}$. We describe the deformation of the entire body in time by the function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t).$$

The function $\mathbf{x}(\mathbf{X}, t)$ gives the place occupied by the material particle $\mathbf{X}$ at time t. The inverse function, $\mathbf{X}(\mathbf{x}, t)$, tells us which material particle is at place $\mathbf{x}$ at time t.

The formulation in the previous pages uses the material coordinate $\mathbf{X}$ and time t as independent variables, a formulation known as the Lagrangian formulation. The formulation results in initial boundary value problems that evolve in time various fields with coordinates of material particles in the reference state. Here we develop the formulation using the spatial coordinate $\mathbf{x}$, known as the Eulerian formulation.

Time derivative of a function of material particle. Let Q be a physical quantity. For example, Q can be the temperature. The function $Q = f(\mathbf{X}, t)$ represents the temperature of material particle $\mathbf{X}$ at time t. The function $Q = g(\mathbf{x}, t)$ represents the temperature of the material particle at place $\mathbf{x}$ at time t. The two functions are related as

$$f(\mathbf{X}, t) = g(\mathbf{x}, t), \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

The rate of change in temperature of the material particle is

$$\frac{\partial f(\mathbf{X}, t)}{\partial t}.$$

This rate is known as the material time derivative. We can calculate the material time derivative by using the function $g(\mathbf{x}, t)$. Using chain rule, we obtain that

$$\frac{\partial f(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i(\mathbf{X}, t)}{\partial t}.$$

Recall that the velocity of the material particle $\mathbf{X}$ at time t is

$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}.$$

Thus, we can calculate the substantial time rate from

$$\frac{\partial f(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} v_i(\mathbf{x}, t).$$

In the above, we have used three symbols to represent the temperature: Q, f and g. This practice is impractical when we deal with many different quantities.

We will use one symbol to represent the same quantity: $Q(\mathbf{X},t)$ and $Q(\mathbf{x},t)$. They represent the temperature as two functions different independent variables. Using this notation, the above change of variable is written as

$$\frac{\partial Q(\mathbf{X},t)}{\partial t} = \frac{\partial Q(\mathbf{x},t)}{\partial t} + \frac{\partial Q(\mathbf{x},t)}{\partial x_i} v_i(\mathbf{x},t).$$

In the Lagrangean formulation, the acceleration is

$$\mathbf{a} = \frac{\partial \mathbf{v}(\mathbf{X},t)}{\partial t}.$$

In the Eulerian formulation, the acceleration of a material particle is

$$a_i(\mathbf{x},t) = \frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial x_j} v_j(\mathbf{x},t).$$

Rate of deformation. Let $\mathbf{v}(\mathbf{x},t)$ be the velocity field. Let $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ be the places occupied by two material particles when the body is in the current state at time *t*. The two material particles are the ends of a straight segment. The vector $\mathbf{v}(\mathbf{x}+d\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)$ is the rate at which the straight segment changes. The Taylor expansion gives

$$v_i(\mathbf{x}+d\mathbf{x},t) - v_i(\mathbf{x},t) = \frac{\partial v_i(\mathbf{x},t)}{\partial x_j} dx_j$$

The velocity gradient is

$$L_{ij} = \frac{\partial v_i(\mathbf{x},t)}{\partial x_j}.$$

The rate of deformation is

$$D_{ij} = \frac{1}{2}\left[\frac{\partial v_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial v_j(\mathbf{x},t)}{\partial x_i}\right].$$

The vorticity is

$$W_{ij} = \frac{1}{2}\left[\frac{\partial v_i(\mathbf{x},t)}{\partial x_j} - \frac{\partial v_j(\mathbf{x},t)}{\partial x_i}\right].$$

Conservation of mass. In the Lagrange formulation, the **nominal mass density** is defined by

$$\rho_R = \frac{\text{mass in current state}}{\text{volume in reference state}}.$$

That is, $\rho_R dV$ is the mass of a material element of volume. A subscript is added here to remind us that the volume is in the reference state.

In the Eularian formulation, the **true mass density** is defined by

$$\rho = \frac{\text{mass in current state}}{\text{volume in current state}}.$$

That is, ρdv is the mass of a spatial element of volume.

The two definitions of density are related as

$$\rho_R dV = \rho dv,$$

or

$$\rho_R = \rho \det \mathbf{F}.$$

The conservation of mass requires that the mass of the material element of volume be time-independent. Thus, the nominal density can only vary with material particle, $\rho_R(\mathbf{X})$, and is time-independent. By contrast, the true density is a function of both place and time, $\rho(\mathbf{x},t)$. The conservation of mass requires that

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho(\mathbf{x},t) v_i(\mathbf{x},t)\right] = 0.$$

When the material is incompressible, $\det \mathbf{F} = 1$, we obtain that

$$\rho_R(\mathbf{X}) = \rho(\mathbf{x},t).$$

Balance of momentum. The true stress obeys that

$$\frac{\partial \sigma_{ij}(\mathbf{x},t)}{\partial x_j} + b_i(\mathbf{x},t) = \rho(\mathbf{x},t)\left[\frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial x_j} v_j(\mathbf{x},t)\right],$$

in the volume of the body, and

$$\sigma_{ij} n_j = t_i$$

on the surface of the body. These are familiar equations used in fluid mechanics.

REFERENCES

Recent Books on Finite Elasticity and Continuum Mechanics

S.S. Antman, Nonlinear Problems of Elasticity, 2nd ed. Springer, 2005.

B. Audoly and Y. Pomeau, Elasticity and Geometry. Oxford University Press, 2010.

T. Belytschko, W.K. Liu and B. Moran, Nonlinear Finite Elements for Continua and Structures. Wiley, 2000.

A. Bertram, Elasticity and Plasticity of Large Deformation. 3rd ed. Springer, 2012.

D. Bigoni, Nonlinear Solid Mechanics. Cambridge University Press, 2012.

J. Bonet, R.D. Wood, Nonlinear Continuum Mechanics for Finite Element Analysis, 2nd ed. Cambridge University Press, 2008.

A.F. Bower, Applied Mechanics of Solids. The full text is free online: http://solidmechanics.org/

F.M. Capaldi. Continuum Mechanics. Cambridge University Press, 2012.

O. Gonzalez and A.M. Stuart. A First Course in Continuum Mechanics. Cambridge University Press, 2008.

M.E. Gurtin, E. Fried and Lallit Anand, The Mechanics and Thermodynamics of Continua. Cambridge University Press, 2010.

G.A. Holzapfel, Nonlinear Solid Mechanics. Wiley, 2000.

P. Howell, G. Kozyrefe, J. Ockendon, Applied Solid Mechanics. Cambridge University Press, 2009.

F. Irgens, Continuum Mechanics. Springer, 2008.

V.A. Lubarda, Elastoplasticity Theory. CRC Press, 2002.

G.A. Maugin, Continuum Mechanics through the Twenties Century. Springer, 2013.

S. Nair, Introduction to Continuum Mechanics. Cambridge University Press, 2009.

J.N. Reddy, An Introduction to Continuum Mechanics. Cambridge University Press, 2008.

A.A. Shabana, Computational Continuum Mechanics. Cambridge University Press, 2008.

E. Tadmor, R.F. Miller and R.S. Elliott, Continuum Mechanics and Thermodynamics. Cambridge University Press, 2012.

C. Trusdell and W. Noll, The Non-linear Field Theories of Mechanics, 3rd edition, Springer, 2004.

J.L. Wegner and J.B. Haddow, Elements of continuum mechanics and thermodynamics. Cambridge University Press, 2009.

匡震邦, 非线性连续介质力学, 上海交通大学出版社, 2002. Written by the man who taught me the subject in 1985 in Xian Jiaotong University.

黄筑平，连续介质力学基础, 2nd ed, 高等教育出版社，2012.